Cartan-Karlhede algorithm and Cartan invariants for spacetimes III

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Outline

From spinors to NP tetrads

For an arbitrary spin basis, (o, ι), such that $[o, \iota] = 1$ then we can consider $Sp(1)$ transformations to construct a new spin basis

1. $(o', \iota') = (\lambda o, \lambda^{-1}\iota), \ \lambda = ae^{i\theta}, \lambda, \theta \in \mathbb{R}.$'
2. $(o', \iota') = (o, \iota + bo), \ \ b \in \mathbb{C}.$
3. $(o', \iota') = (o + c\iota, \iota), \ \ c \in \mathbb{C}.$

In particular, using the Petrov classification for Weyl spinors, we can use these transformations to align the spin frame with the Weyl spinor.

Using the Infeld-van der Waerden symbols we can relate this to a NP frame $\{\ell, n, m, \bar{m}\}$:

$$\begin{aligned} \ell^a = o^A\bar{o}^{A'}, \ n^a = \iota_A\bar{\iota}^{A'}, \ m^a = o^A\bar{\iota}^{A'}, \ \bar{m}^a = \iota^A\bar{o}^{A'}, \\ \ell_a = o_A\bar{o}_{A'}, \ n_a = \iota_A\bar{\iota}_{A'}, \ m_a = o_A\bar{\iota}_{A'}, \ \bar{m}_a = \iota_A\bar{o}_{A'}, \end{aligned} \tag{1}$$

Lorentz frame transformations

For the NP basis $\{\ell, n, m, \bar{m}\}$ where

$$g_{ab} = 2\ell_{(a}n_{b)} - m_{(a}\bar{m}_{b)}, \tag{2}$$

the Lorentz frame transformation group is then:

- Boosts and Spins:

$$\ell' = a^2\ell, \quad n' = a^{-2}n, m' = e^{2i\theta}m. \tag{3}$$

- Null rotations about ℓ:

$$\ell' = \ell, \quad n' = n + bm + \bar{b}\bar{m} + |b|^2\ell, m' = m + \bar{b}\ell. \tag{4}$$

- Null rotations about n:

$$n' = n, \quad \ell' = \ell + cm + \bar{c}\bar{m} + |c|^2 n, m' = m + \bar{c}n. \tag{5}$$

These are the corresponding transformations we would use to build a frame adapted to the Petrov classification of the Weyl tensor.

How to differentiate a spinor

If θ, ϕ and ψ are spinor fields defined on M, where θ and ϕ have the same valence. The spinor covariant derivative is defined as a map $\nabla_X = \nabla_{XX'} : \theta_{\dots} \to \theta_{\dots;XX'}$ such that

- $\nabla_X(\theta + \phi) = \nabla_X\theta + \nabla_X\phi$
- $\nabla_X(\theta\psi) = (\nabla_X\theta)\psi + \theta\nabla_X\psi.$
- $\psi = \nabla_X\theta$ implies $\bar{\psi} = \nabla_X\bar{\theta}$
- $\nabla_X\epsilon^{AB} = \nabla_X\epsilon_{AB} = 0$
- ∇_X commutes with any index substitution not involving X or X'
- $\nabla_X\nabla_Y f = \nabla_Y\nabla_X f$ for f a scalar (torsion-free)
- For any derivation D acting on spinor fields, there is a spinor $\zeta^{XX'}$ such that $D\psi = \zeta^{XX'}\nabla_{XX'}\psi$ for all ψ.

This identifies the 4D vector space of Hermitian spinors $\tau^{AA'}$ with $T_p(M)$ and the dual vector space with $T_p^*(M)$

Good news: ∇_X exists and is unique [Penrose and Rindler 1984, section 4.4].

Detour into frame fields

The tetrad formalism is one way to compute the curvature tensor.

Suppose that $e^i{}_a$ is a tetrad of vectors, with corresponding dual e_i^a, so that

$$e_i{}^a e^i{}_b = \delta_a{}^b.$$

i, j, k, l label the (co)vectors.

a, b, c, d label the components with respect to some arbitrary chosen basis.

The Ricci rotation coefficients are then

$$\Gamma_{ijk} = e_i{}^a e_k{}^b \nabla_b e_j{}^a = -e_j{}^a e_k{}^b \nabla_b e_i{}^a. \quad (6)$$

where ∇ is the Levi-Civita connection for tensors.

From the Ricci identity we have

$$R_{abcd} = 2 e_{ia} \nabla_{[c} \nabla_{d]} e^i{}_b. \quad (7)$$

Spinor dyads

Introduce the spinor dyad $\epsilon_I{}^A$ and its symplectic dual $\epsilon^I{}_A$ so that

$$\epsilon_0{}^A = o^A,\ \epsilon_1{}^A = \iota^A,\ \epsilon_I{}^A \epsilon^I{}_B = \epsilon^A{}_B.$$

Then **the spinor Ricci rotation coefficients** are

$$\Gamma_{IJKK'} = \epsilon_{IA} \epsilon_K^C \epsilon_{K'}^{C'} \nabla_{CC'} \epsilon_J^A \tag{8}$$

The spinor equivalent of the curvature tensor is then

$$\begin{aligned} R_{ABCDA'B'C'D'} &= 2\epsilon_{IA}\epsilon_{I'A'} \nabla_{[c}\nabla_{d]}(\epsilon^I{}_B \epsilon^{I'}{}_{B'}) \\ &= 2\epsilon_{IA}\epsilon_{I'A'} \epsilon^{I'}{}_{B'} \nabla_{[c}\nabla_{d]}\epsilon^I{}_B + \text{c.c.} \\ &= 2\epsilon_{IA}\epsilon_{A'B'} \nabla_{[c}\nabla_{d]}\epsilon^I{}_B + \text{c.c.} \end{aligned} \tag{9}$$

Here, $\nabla_c = \nabla_{CC'}$ and $\nabla_d = \nabla_{DD'}$ and this can be rewritten as

$$R_{ABCDA'B'C'D'} = \epsilon_{IA}\epsilon_{A'B'}(\epsilon_{C'D'}\Box_{CD}\epsilon^I{}_B + \epsilon_{CD}\Box_{C'D'}\epsilon^I{}_B) + \text{c.c.}$$

where $\Box_{CD} = \nabla_{C'(C}\nabla_{D)}^{C'}$ and $\Box_{C'D'} = \nabla_{C(C'}\nabla_{D')}^{C}$.

Curvature spinor

The first term $\epsilon_{IA}\Box_{CD}\epsilon^{I}{}_{B}$ is symmetric in CD and AB.

This tensor can be decomposed as

$$\epsilon_{IA}\Box_{CD}\epsilon^{I}{}_{B} = \Psi_{ABCD} - 2\Lambda\epsilon_{(A(C}\epsilon_{D)B)}. \tag{10}$$

where

$$\Psi_{ABCD} = \epsilon_{IA}\Box_{(CD}\epsilon^{I}{}_{B)}, \quad \Lambda = \frac{1}{6}\epsilon_{IA}\Box^{AB}\epsilon^{I}{}_{B}. \tag{11}$$

Similarly, the second term can be written as

$$\epsilon_{IA}\Box_{C'D'}\epsilon^{I}{}_{B} = \Phi_{ABC'D'} \tag{12}$$

which is symmetric in AB and $C'D'$.

Thus the curvature spinor can be written as

$$R_{ABCDA'B'C'D'} = \epsilon_{A'B'}\epsilon_{C'D'}[\Psi_{ABC} - 2\Lambda\epsilon_{(A(C)}\epsilon_{D)B)}] + \epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} + \text{c.c.}. \tag{13}$$

Components of the curvature spinor

$$R_{ABCDA'B'C'D'} = \epsilon_{A'B'}\epsilon_{C'D'}[\Psi_{ABC} - 2\Lambda\epsilon_{(A(C)}\epsilon_{D)B)}] + \epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} + \text{c.c.} \quad (14)$$

From $R_{a[bcd]} = 0$, it follows that $\Lambda \in \mathbb{R}$ and $\Phi_{ABA'B'}$ is Hermitian.

Contracting two indices, we have

$$R_{ABA'B'} = -2\Phi_{ABA'B'} + 6\Lambda\epsilon_{AB}\epsilon_{A'B'} \leftrightarrow R_{ab} = -2\Phi_{ab} + 6\Lambda g_{ab} \quad (15)$$

and so we recover the Ricci scalar and the trace-free Ricci tensor:

$$\Lambda = \frac{R}{24}, \quad \Phi_{ab} = -\frac{1}{2}\left(R_{ab} - \frac{1}{4}Rg_{ab}\right). \quad (16)$$

The remaining term is a Hermitian spinor which gives the Weyl tensor

$$\Psi_{ABC}\epsilon_{A'B'}\epsilon_{C'D'} + \text{c.c.} \quad (17)$$

The differential Bianchi identities, $R_{ab[cd;e]} = 0$ give

$$\nabla^{D}{}_{C'}\Psi_{ABCD} = \nabla^{D'}{}_{(C}\Phi_{AB)C'D'}, \quad \nabla^{BB'}\Phi_{ABA'B'} = -3\nabla_{AA'}\Lambda. \quad (18)$$

Spin coefficients

Rewrite quantities explicitly using a particular NP frame and associated derivatives:

$$D = \ell^a \nabla_a, \;\; \Delta = n^a \nabla_a, \delta = m^a \nabla_a, \; \bar{\delta} \nabla_a \tag{19}$$

Then the (spinor) Ricci rotation coefficients can be written down as 12 complex-valued scalars:

$$\begin{array}{|c|c|c|c|} \nabla_{BB'} & o^A \nabla_{BB'} o_A & o^A \nabla_{BB'} \iota_A = \iota^A \nabla_{BB'} o_A & \iota^A \nabla_{BB'} \iota_A \\ \nabla_b & m^a \nabla_b \ell_a & \frac{1}{2}(n^a \nabla_b \ell_a - \bar{m}^a \nabla_b m_a) & -\bar{m}^a \nabla_b n_a \\ D & \kappa & \epsilon & \pi \\ \Delta & \tau & \gamma & \nu \\ \delta & \sigma & \beta & \mu \\ \bar{\delta} & \rho & \alpha & \lambda \end{array} \tag{20}$$

These are known as the NP **spin coefficients**.

As $\nabla_{BB'}$ is torsion free, we can also write down the commutators for the derivations.

Curvature scalars

Using the spinor dyad (o^A, ι^A) we can also write down the NP **curvature scalars**:

$$\Phi_{00} = \Phi_{ABA'B'}\, o^A o^B \bar{o}^{A'} \bar{o}^{B'} ,\ \Phi_{11} = \Phi_{ABA'B'}\, o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} ,\ \Phi_{22} = \Phi_{ABA'B'}\, \iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} ,$$
$$\Phi_{01} = \Phi_{ABA'B'}\, o^A o^B \bar{o}^{A'} \bar{\iota}^{B'} ,\ \Phi_{10} = \Phi_{ABA'B'}\, o^A \iota^B \bar{o}^{A'} \bar{o}^{B'} ,$$
$$\Phi_{02} = \Phi_{ABA'B'}\, o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'} ,\ \Phi_{20} = \Phi_{ABA'B'}\, \iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'} ,$$
$$\Phi_{12} = \Phi_{ABA'B'}\, o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} ,\ \Phi_{21} = \Phi_{ABA'B'}\, \iota^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} .$$

$$\Psi_0 = \Psi_{ABCD} o^A o^B o^C o^D,\ \Psi_1 = \Psi_{ABCD} o^A o^B o^C \iota^D,$$
$$\Psi_2 = \Psi_{ABCD} o^A o^B \iota^C \iota^D,\ \Psi_3 = \Psi_{ABCD} o^A \iota^B \iota^C \iota^D,$$
$$\Psi_4 = \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D.$$

We could write down the Ricci equations, $R_{abcd} = 2 e_{ia} \nabla_{[c} \nabla_{d]} e^i{}_b$, and Bianchi identities, $R_{ab[cd;e]} = 0$ in terms of these quantities.

As an example, we will consider the vacuum type N spacetimes [Collins, 1991], so that

$$\Lambda = 0, \ \Phi_{ABA'B'} = 0 \text{ and } \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$$

according to the Petrov classification.

This is done using a rotation to fix the principal spinor $\alpha^A = o^A$.

We can also employ a Lorentz transformations to set $\Psi_4 = 1$.

The Bianchi identities, $R_{ab[cd;e]}$ are then

$$\begin{aligned} \kappa &= 0, \\ \sigma &= 0, \\ 4\epsilon &= \rho, \\ 4\beta &= \tau \end{aligned} \tag{21}$$

NP "field" equations

The Ricci equations, $R_{abcd} = 2e_{ia}\nabla_{[c}\nabla_{d]}e^{i}{}_{b}$, are

$$D\rho = \tfrac{5}{4}\rho^2 + \tfrac{1}{4}\rho\bar{\rho} \tag{2.4a}$$

$$D\tau = \tfrac{5}{4}\rho\tau + \bar{\pi}\rho - \tfrac{1}{4}\bar{\rho}\tau \tag{2.4b}$$

$$D\alpha - \tfrac{1}{4}\bar{\delta}\rho = \tfrac{1}{2}\rho\alpha + \tfrac{1}{4}\bar{\rho}\alpha - \tfrac{1}{16}\bar{\tau}\rho + \tfrac{5}{4}\rho\pi \tag{2.4c}$$

$$D\gamma - \tfrac{1}{4}\Delta\rho = \tfrac{5}{4}\tau\pi - \tfrac{1}{2}\gamma\rho + \tau\alpha + \alpha\bar{\pi} + \tfrac{1}{4}\tau\bar{\tau} - \tfrac{1}{4}\bar{\gamma}\rho - \tfrac{1}{4}\gamma\bar{\rho} \tag{2.4d}$$

$$D\lambda - \bar{\delta}\pi = \tfrac{1}{4}\rho\lambda + \tfrac{1}{4}\bar{\rho}\lambda + \pi^2 + \alpha\pi - \tfrac{1}{4}\bar{\tau}\pi \tag{2.4e}$$

$$D\mu - \delta\pi = \tfrac{3}{4}\bar{\rho}\mu + \pi\bar{\pi} - \tfrac{1}{4}\rho\mu - \pi\bar{\alpha} + \tfrac{1}{4}\pi\tau \tag{2.4f}$$

$$D\nu - \Delta\pi = \pi\mu + \bar{\tau}\mu + \bar{\pi}\lambda + \tau\lambda + \gamma\pi - \bar{\gamma}\pi - \tfrac{3}{4}\rho\nu - \tfrac{1}{4}\bar{\rho}\nu \tag{2.4g}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \bar{\mu}\lambda - 3\gamma\lambda + \bar{\gamma}\lambda + 3\alpha\nu + \pi\nu - \tfrac{3}{4}\bar{\tau}\nu - \Psi_4 \tag{2.4h}$$

$$\delta\rho = \tfrac{5}{4}\tau\rho + \bar{\alpha}\rho - \bar{\rho}\tau \tag{2.4i}$$

$$\delta\alpha - \tfrac{1}{4}\bar{\delta}\tau = \tfrac{5}{4}\mu\rho + \alpha\bar{\alpha} + \tfrac{1}{16}\tau\bar{\tau} - \tfrac{1}{2}\alpha\tau + \gamma\rho - \gamma\bar{\rho} - \tfrac{1}{4}\rho\bar{\mu} \tag{2.4j}$$

$$\delta\lambda - \bar{\delta}\mu = \rho\nu - \bar{\rho}\nu + \mu\pi - \bar{\mu}\pi + \mu\alpha + \tfrac{1}{4}\mu\bar{\tau} + \lambda\bar{\alpha} - \tfrac{3}{4}\lambda\tau \tag{2.4k}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu - \bar{\nu}\pi + \tfrac{1}{4}\tau\nu - \bar{\alpha}\nu \tag{2.4l}$$

$$\delta\gamma - \tfrac{1}{4}\Delta\tau = \tfrac{1}{2}\tau\gamma - \bar{\alpha}\gamma + \tfrac{5}{4}\mu\tau - \tfrac{1}{4}\rho\bar{\nu} + \tfrac{1}{4}\tau\bar{\gamma} + \alpha\bar{\lambda} \tag{2.4m}$$

$$\delta\tau = \bar{\lambda}\rho + \tfrac{5}{4}\tau^2 - \tau\bar{\alpha} \tag{2.4n}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \tfrac{3}{4}\bar{\tau}\tau - \alpha\tau + \gamma\rho + \bar{\gamma}\rho \tag{2.4o}$$

$$\Delta\alpha - \bar{\delta}\gamma = \tfrac{5}{4}\rho\nu - \tfrac{5}{4}\tau\lambda + \bar{\gamma}\bar{\alpha} - \bar{\mu}\bar{\alpha} - \tfrac{3}{4}\tau\gamma. \tag{2.4p}$$

Figure: Taken from [Collins, 1991]

Type N subclasses

- pp-wave metrics

$$\rho = 0, \quad \tau = 0.$$

- Rotating plane-fronted wave metrics (Kundt waves)

$$\rho = 0, \quad \tau \neq 0.$$

- Robinson-Trautman metrics

$$\rho \neq 0, Im(\rho) = 0.$$

- The twisting case

$$\rho \neq 0, Im(\rho) \neq 0.$$

So far, only the Hauser metric is the sole example of vacuum type N spacetimes with twist.

The Cartan-Karlhede Algorithm

Denote the set of components $\{R_{abcd}, R_{abcd;e_1}, \ldots e_q\}$ as R^q.

The algorithm is then:

1. Let $q = 0$.
2. Compute R^q.
3. Fix the frame as much as possible using Lorentz frame transformations.
4. Find the invariance group H^q of the frame which leaves R^q invariant.
5. Find the number of functionally independent components t^q amongst the set R^q.
6. If $t^q \neq t^{q-1}$ or $dim(H^q) \neq dim(H^{q-1})$ then set $q = q + 1$ and go to step 2. Otherwise, the algorithm stops and set $q = p + 1$.

The set $\{H^r, t^r, R^r\}$, $r = 1, ..., p + 1$ classifies the solution, locally.

Definition

The set R^p relative to the frame basis determined by the Cartan-Karlhede algorithm are called *Cartan invariants*.

First order derivatives

$$
\begin{aligned}
(D\Psi)_{\hat{\mu}E'} &= \Psi_{ABCD;EE'}\epsilon_I{}^A\epsilon_J{}^B\epsilon_K{}^C\epsilon_L{}^D\epsilon_M^{\ E}\epsilon_{M'}^{\ E'} \\
&= (\Psi_{ABCD}\epsilon_I{}^A\epsilon_J{}^B\epsilon_K{}^C\epsilon_L{}^D)_{;EE'}\epsilon_M^{\ E}\epsilon_{M'}^{\ E'} - (\epsilon_I{}^A\epsilon_J{}^B\epsilon_K{}^C\epsilon_L{}^D)_{EE'}\epsilon_M^{\ E}\epsilon_{M'}^{\ E'}\Psi_{ABCD},
\end{aligned} \tag{22}
$$

where, $\hat{\mu}$ counts the appearance of ι. There are 3 cases:

$$
\begin{aligned}
\hat{\mu} = 5 : (D\Psi)_{\hat{\mu}E'} &= (\Psi_4)_{;1E'} - 4\Gamma_{111E'}\Psi_3 + 4\Gamma_{101E'}\Psi_4, \\
\hat{\mu} = 4 : (D\Psi)_{\hat{\mu}E'} &= (\Psi_4)_{,0E'} - 4\Gamma_{110E'}\Psi_3 + 4\Gamma_{100E'}\Psi_4, \\
\hat{\mu} < 4 : (D\Psi)_{\hat{\mu}E'} &= (\Psi_{\hat{\mu}})_{;0E'} - \hat{\mu}\Gamma_{1110E'}\Psi_{\hat{\mu}-1} + \hat{\mu}\Gamma_{100E'}\Psi_{\hat{\mu}} - (2\hat{\mu}-4)\Gamma_{100E'}\Psi_{\hat{\mu}} \\
&\quad + (4-\hat{\mu})\Gamma_{000E'}\Psi_{\hat{\mu}+1}.
\end{aligned} \tag{23}
$$

In the vacuum type N spacetimes we find:

$$
\begin{aligned}
(D\Psi)_{40'} &= \rho, \\
(D\Psi)_{50'} &= 4\alpha, \\
(D\Psi)_{41'} &= \tau, \\
(D\Psi)_{51'} &= 4\gamma.
\end{aligned} \tag{24}
$$

Second derivatives

$$(D^2\Psi)_{\hat{\mu}E';FF'} = \Psi_{ABCD;EE';FF'}[\epsilon_I{}^A\epsilon_J{}^B\epsilon_K{}^C\epsilon_L{}^D\epsilon_M{}^E]\epsilon_{M'}{}^{E'}\epsilon_N{}^F\epsilon_{N'}{}^{F'}, \tag{25}$$

or with the Leibnitz rule...

$$\begin{aligned}(D^2\Psi)_{\hat{\mu}E';FF'} = & (\Psi_{ABCD;EE'}\epsilon_I{}^A\epsilon_J{}^B\epsilon_K{}^C\epsilon_L{}^D\epsilon_M{}^E]\epsilon_{M'}{}^{E'})_{;FF'}\epsilon_N{}^F\epsilon_{N'}{}^{F'} \\ & - (\epsilon_I{}^A\epsilon_J{}^B\epsilon_K{}^C\epsilon_L{}^D\epsilon_M{}^E]\epsilon_{M'}{}^{E'})_{;FF'}\epsilon_N{}^F\epsilon_{N'}{}^{F'}\Psi_{ABCD;EE'}.\end{aligned} \tag{26}$$

With some work, this can be written as

$$\begin{aligned}(D^2\Psi)_{\hat{\mu}E';FF'} = & [(D\Psi)_{\hat{\mu}E'}]_{;FF'} - \hat{\mu}\Gamma_{11FF'}(D\Psi)_{(\hat{\mu}-1)E'} + (2\hat{\mu}-5)\Gamma_{10FF'}(D\Psi)_{\hat{\mu}E'} \\ & + (5-\hat{\mu})\Gamma_{00FF'}(D\Psi)_{(\hat{\mu}+1)E'} - \bar{\Gamma}_{E'1'F'F}(D\Psi)_{\hat{\mu}0'} + \bar{\Gamma}_{E'0'F'F}(D\Psi)_{\hat{\mu}1'}.\end{aligned} \tag{27}$$

Vacuum type N spacetimes

$$(D^2\Psi)_{30';10'} = 2\rho^2,$$
$$(D^2\Psi)_{30';11'} = 2\rho\tau,$$
$$(D^2\Psi)_{31';10'} = 2\rho\tau,$$
$$(D^2\Psi)_{31';11'} = 2\tau^2,$$
$$(D^2\Psi)_{40';00'} = D\rho + \frac{3}{4}\rho^2 - \frac{1}{4}|\rho|^2,$$
$$(D^2\Psi)_{40';10'} = \bar{\delta}\rho + 7\alpha\rho - \frac{1}{4}\bar{\tau}\rho,$$
$$(D^2\Psi)_{40';01'} = \delta\rho + \frac{3}{4}\tau\rho - \bar{\alpha}\rho + \tau\bar{\rho},$$
$$(D^2\Psi)_{40';11'} = \Delta\rho + 3\gamma\rho + 4\alpha\tau - \bar{\gamma}\rho + |\tau|^2,$$
$$(D^2\Psi)_{41';00'} = D\tau + \frac{3}{4}\rho\tau - \bar{\pi}\rho + \frac{1}{4}\bar{\rho}\tau,$$
$$(D^2\Psi)_{41';10'} = \bar{\delta}\tau + 3\alpha\tau + 4\gamma\rho - \bar{\mu}\rho + \frac{1}{4}|\tau|^2,$$

Vacuum type N spacetimes

$$
\begin{aligned}
(D^2\Psi)_{41';01'} &= \delta\tau + \frac{3}{4}\tau^2 - \bar{\lambda}\rho + \bar{\alpha}\tau,\\
(D^2\Psi)_{41';11'} &= \Delta\tau + 7\gamma\tau - \bar{\nu}\rho + \bar{\gamma}\tau,\\
(D^2\Psi)_{50';00'} &= 4D\alpha - 5\pi\rho + 5\alpha\rho - \bar{\rho}\alpha,\\
(D^2\Psi)_{50';10'} &= 4\bar{\delta}\alpha - 5\lambda\rho + 20\alpha^2 - \bar{\tau}\alpha,\\
(D^2\Psi)_{50';01'} &= 4\delta\alpha - 5\mu\rho + 5\tau\alpha - 4|\alpha|^2 + 4\gamma\bar{\rho},\\
(D^2\Psi)_{50';11'} &= 4\Delta\alpha - 5\nu\rho + 20\gamma\alpha - 4\bar{\gamma}\alpha + 4\gamma\bar{\tau},\\
(D^2\Psi)_{51';00'} &= 4D\gamma - 5\pi\tau + 5\rho\gamma - 4\bar{\pi}\alpha + \bar{\rho}\gamma,\\
(D^2\Psi)_{51';10'} &= 4\bar{\delta}\gamma - 5\lambda\tau + 20\alpha\gamma - 4\bar{\mu}\alpha + \bar{\tau}\gamma,\\
(D^2\Psi)_{51';01'} &= 4\delta\gamma - 5\mu\tau + 5\gamma\tau - 4\bar{\lambda}\alpha + 4\gamma\bar{\alpha},\\
(D^2\Psi)_{51';11'} &= 4\Delta\gamma - 5\nu\tau + 20\gamma^2 - 4\bar{\nu}\alpha + 4|\gamma|^2.
\end{aligned}
$$

Linear Isotropy

We see that four spin-coefficients appear at first order:

$$\rho, \alpha, \tau, \text{ and } \gamma, \tag{28}$$

along with their derivatives at higher orders.

At zeroth order, we can transform the spin frame and leave the Type N condition unchanged:

$$o' = o, \iota' = \iota + bo \tag{29}$$

Under this transformation, the "first order" spin-coefficients above transform as

$$\begin{aligned} \rho' &= \rho, \\ \alpha' &= \alpha + \frac{5}{4}\bar{b}\rho, \\ \tau' &= \tau + b\rho, \\ \gamma' &= \gamma + b\alpha + \frac{5}{4}\bar{b}\tau + \frac{5}{4}|b|^2\rho. \end{aligned} \tag{30}$$

Summary of Collins' analysis

	I	*IIa*	*IIb*	*IIIa*	*IIIb*
Invariant characterization	$\rho \neq 0$	$\rho = 0$ $\tau = 0$ $\alpha \neq 0$	$\rho = 0$ $\tau = 0$ $\alpha = 0$	$\rho = 0$ $\tau \neq 0$ $\lvert\alpha\rvert \neq \frac{5}{4}\lvert\tau\rvert$	$\rho = 0$ $\tau \neq 0$ $\lvert\alpha\rvert = \frac{5}{4}\lvert\tau\rvert$
Canonical form:					
Zeroth order	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$
First order	$\tau = 0$	$\gamma = 0$		$\gamma = 0$	$Re(\gamma)$ or $Im(\gamma) = 0$
Second order					$Re(\Delta\tau) = 0$
Upper bound	5	4	2	~~5~~ 4	~~6~~ ~~5~~ 3

- In case III, Collins (1991) provided an upper-bound of 5 and 6 for a and b respectively.
- Ramos and Vickers (1996) using the GHP formalism gave an upper-bound of 5 for III [Ramos and Vickers, 1996]
- DDM. Milson and Coley (2013) lowered the upper-bounds and provided examples, showing they were sharp [McNutt et al., 2013].

The Cartan-Karlhede Algorithm

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The algorithm is then:

1. Let $q = 0$.
2. Compute R^q.
3. Fix the frame as much as possible using Lorentz frame transformations.
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6. If $t^q \neq t^{q-1}$ or $dim(H^q) \neq dim(H^{q-1})$ then set $q = q + 1$ and go to step 2. Otherwise, the algorithm stops and set $q = p + 1$.

The set $\{H^r, t^r, R^r\}$, $r = 1, ..., p + 1$ classifies the solution, locally.

Definition

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4D: Higher rank spinors/tensors

We can treat the Weyl tensor, C_{abcd} and the Ricci tensor, R_{ab}, as operators on some vector space and fin canonical forms of the operators.

For example, the self dual Weyl tensor

$$C^*{}_{abcd} = C_{abcd} + i\frac{1}{2}C_{ab}{}^{ef}\epsilon_{efcd} \tag{31}$$

can be seen as an operator acting on the 6-dimensional space of self dual bivectors, $X^*_{ab} = -i\frac{1}{2}\epsilon_{abcd}X^{*cd}$:

$$C^*{}_{ab}{}^{cd}X^*_{cd} = Y^*_{ab} \tag{32}$$

Can we treat $C^*{}_{abcd;e}$ or $R_{ab;e}$ as operators on some vector space?

In general, no.

Recall the Weyl spinor

$$\Psi = \Psi_{IJKL}\epsilon^{I}{}_{A}\epsilon^{J}{}_{B}\epsilon^{K}{}_{C}\epsilon^{L}{}_{D} \tag{33}$$

We can count the number of principal spinors, o^A and relate these to principal null directions of the self-dual tensor

$$C^{*}_{abcd} = 2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} \tag{34}$$

To count the number of appearance of o^a in each term, we can consider a boost:

$$o' = ao, \;\; \iota' = a^{-1}\iota, \leftrightarrow \ell' = a^2\ell, \;\; n' = a^{-2}n \tag{35}$$

In this representation, we find that

$$\Psi_0' = a^4\Psi_0, \;\; \Psi_1' = a^2\Psi_1, \;\; \Psi_2' = \Psi_2, \;\; \Psi_3' = a^{-2}\Psi_3, \;\; \Psi_4' = a^{-4}\Psi_4. \tag{36}$$

More generally, consider a boost for an arbitrary tensor [Milson et al., 2005],

$$T'_{a_1 a_2 \dots a_n} = \lambda^{b_{a_1 a_2 \cdots a_n}} T_{a_1 a_2 \dots a_n}, . \tag{37}$$

The quantity,

$$b_{a_1 a_2 \dots a_n} = \sum_{i=1}^{n} (\delta_{a_i 0} - \delta_{a_i 1})$$

is called the *boost weight* (b.w) of the frame component $T_{a_1 a_2 \dots a_p}$.

We can write the tensor **T** in the following decomposition:

$$\mathbf{T} = \sum_{b} (\mathbf{T})_b . \tag{38}$$

$(\mathbf{T})_b$ denotes the projection onto the subspace of components of boost weight b.

Alignment classification

For any **T** we can pick an NP frame and decompose into b.w.:

$$\mathbf{T} = \sum_b (\mathbf{T})_b. \tag{39}$$

Boost order, $\mathcal{B}_{\mathbf{T}}(\ell)$, is the maximum b.w. of a tensor, **T**, for a null direction ℓ.

For a given null direction ℓ, $\mathcal{B}_{\mathbf{T}}(\ell)$ is invariant under boosts, spatial rotations and null rotations about ℓ.

$\mathcal{B}_{\mathbf{T}}(\ell)$ is only dependent on the choice of ℓ.

Defining

$$B_{\mathbf{T}} = \max_{\ell} \mathcal{B}_{\mathbf{T}}(\ell)) \tag{40}$$

the existence of a ℓ with $\mathcal{B}_{\mathbf{T}}(\ell) < B_{\mathbf{T}}$ is an invariant property of the tensor **T**.

We will say ℓ is **T**-aligned if $\mathcal{B}_{\mathbf{T}}(\ell) < B_{\mathbf{T}}$.

To determine the canonical form of R_{abcd}, consider the effect of a boost on its irreducible parts:

$$\begin{array}{ccc} \text{Boost order} & \text{Weyl} & \text{Ricci} \\ 2 & \Psi_0 & R_{00} \\ 1 & \Psi_1 & R_{0i},\ i=3,4 \\ 0 & \Psi_2 & R_{01}, R_{ij},\ i,j=3,4 \\ -1 & \Psi_3 & R_{1i} \\ -2 & \Psi_4 & R_{11} \end{array} \tag{41}$$

Alignment types of the Weyl tensor, C_{abcd}, and Ricci tensor, R_{ab}, are

$$\begin{array}{cccccc} \mathit{Type} & \mathbf{G} & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{N} \\ \mathcal{B}_{\mathbf{T}}(\ell) & 2 & 1 & 0 & -1 & -2. \end{array} \tag{42}$$

If C_{abcd} or R_{ab} vanishes, then it belongs to alignment type **O**.

Alignment type is not enough to reproduce Segre type for R_{ab}, instead we must also examine the algebro-geometric properties of

$$R'_{00} = R_{00} + 2R_{0i}c^i + R_{ij}c^ic^j - R_{01}|c|^2 - R_{1i}c^i|c|^2 + R_{11}|c|^4 = 0. \tag{43}$$

For $R_{abcd;e_1\dots e_p}$, $|\mathcal{B}_{\mathbf{T}}(\ell)|$ may be greater than two but the alignment types are still applicable.

For example, here are the b.w. of the first covariant derivative:

$$\begin{aligned}
b &= -3 \ : C^*{}_{1212;1} = 8\alpha,\\
b &= -2 \ : C^*{}_{0112;1} = C^*{}_{1201;1} = C^*{}_{1212;3} = C^*{}_{1223;2} = C^*{}_{2312;1} = -2\rho,\\
b &= -2 \ : C^*{}_{1212;2} = 8\gamma,\\
b &= -1 \ : C^*{}_{0112;2} = C^*{}_{1201;2} = C^*{}_{1212;0} = C^*{}_{1223;2} = C^*{}_{2312;2} = -2\tau.
\end{aligned} \tag{44}$$

We can consider the transformation rules from before

$$\begin{aligned}
\rho' &= \rho,\\
\alpha' &= \alpha + \frac{5}{4}\bar{b}\rho,\\
\tau' &= \tau + b\rho,\\
\gamma' &= \gamma + b\alpha + \frac{5}{4}\bar{b}\tau + \frac{5}{4}|a|^2\rho.
\end{aligned} \tag{45}$$

Conclusions

	I	IIa	IIb	$IIIa$	$IIIb$
Invariant characterization	$\rho \neq 0$	$\rho = 0$ $\tau = 0$ $\alpha \neq 0$	$\rho = 0$ $\tau = 0$ $\alpha = 0$	$\rho = 0$ $\tau \neq 0$ $\lvert\alpha\rvert \neq \frac{5}{4}\lvert\tau\rvert$	$\rho = 0$ $\tau \neq 0$ $\lvert\alpha\rvert = \frac{5}{4}\lvert\tau\rvert$
Canonical form:					
Zeroth order	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$	$\Psi_i = \delta^4_i$
First order	$\tau = 0$	$\gamma = 0$		$\gamma = 0$	$Re(\gamma)$ or $Im(\gamma) = 0$
Second order					$Re(\Delta\tau) = 0$
Upper bound	5	4	2	4	3

At first order $|\mathcal{B}_{\mathbf{T}}(\ell)|$ is not enough to distinguish some cases

$$\begin{array}{cccccc} \textit{Subclass} & I & IIa & IIb & IIIa & IIIb \\ |\mathcal{B}_{\mathbf{T}}(\ell)| & -2 & -3 & -2 & -1 & -1 \end{array} \tag{46}$$

Coarse canonical forms that can be refined by adding additional geometric conditions.

This is still an open question on how.

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Thank you for your attention!