Dispersionless integrable systems (III)

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Jet formalism

Let $J^\ell M \to M$ be the bundle, whose points are ℓ-jets of functions $u : M \to \mathbb{R}$ (for systems change the target to $\mathbb{R}^m$ or a rank m bundle $\mathcal{V}$ over M). A choice of coordinates x^i on M leads to coordinates (x^i, u_α) on $J^\ell M$, with α being a multi-index of length $|\alpha| \le \ell$. It is important to note that $\pi_{\ell,\ell-1} : J^\ell M \to J^{\ell-1}M$ is an affine bundle modelled on $S^\ell T^* M$.

The infinite jet bundle $J^\infty M$ is a projective limit of $J^\ell M$, and the space of functions on it is the injective limit of $C^\infty(J^\ell M)$. The bundle $J^\infty M$ has a canonical flat connection, the so-called Cartan distribution, for which the horizontal lift

$$\mathcal{D}(M) \ni X \dashrightarrow D_X \in \mathcal{D}(J^\infty M)$$

is characterized by

$$(D_X f) \circ j^\infty u = X(f \circ j^\infty u), \quad \forall f \in C^\infty(J^\infty M), u \in C^\infty(M).$$

In local coordinates, if $X = a^i \partial_i$, then $D_X = a^i D_i$, where $D_i = \partial_i + u_{i\alpha} \partial_{u_\alpha}$ is the operator of total derivative.

Symbols

A (scalar) differential operator of order $\leq \ell$ on M is a function $F \in C^\infty(J^\ell M) \subset C^\infty(J^\infty M)$. It defines a PDE (system) $\mathcal{E} = \{F = 0\} \subset J^\ell M$, as well as its (finite and infinite) prolongation $\mathcal{E}^{(\infty)} = \{D_\alpha F = 0\} \subset J^\infty M$.

The vertical part of the 1-form $dF \in \Omega^1(J^\infty M)$ may be viewed in coordinates as a polynomial on $\pi_\infty^* T^* M$ given by

$$\sum_{j=0}^{\ell} F_{(j)}, \text{ where } F_{(j)} = \sum_{|\alpha|=j} (\partial_{u_\alpha} F)\partial_\alpha \in \Gamma(\pi_\infty^* S^j TM).$$

The top degree $\sigma_F = F_{(\ell)}$ is called the (order ℓ) symbol of F, and at the points of $\mathcal{E}$ it is coordinate-independent.

For instance, for an operator F of the second order,

$$\sigma_F = \sum_{i \leq j} \frac{\partial F}{\partial u_{ij}} \partial_i \partial_j = \sum_{i,j} \sigma_{ij}(u) \partial_i \otimes \partial_j,$$

where $\sigma_{ij}(u) = \frac{1}{2}(1 + \delta_{ij})\partial_{u_{ij}} F$.

This generalizes to PDE systems of order ℓ given as a locus of a function $F : J^{k+\ell}(M,\mathcal{V}) \to J^k(M,\mathcal{W})$ for $k \geq 0$ (prolongation), where $\mathcal{V},\mathcal{W}$ are some (vector) bundles over M.

The symbol σ_F of F is then a homogeneous degree l polynomial on $\pi_\infty^* T^*M$ with values in $\mathrm{Hom}(\mathcal{V},\mathcal{W})$. The characteristic variety of $\mathcal{E} : F = 0$ is defined by

$$\chi^{\mathcal{E}} = \{[\theta] \in \mathbb{P}(\pi_\infty^* T^*M) : \sigma_F(\theta) \text{ is not injective}\}.$$

If $\mathcal{V},\mathcal{W}$ have the same rank ("determined system"), then $[\theta]$ is characteristic iff $\sigma_F(\theta)$ is not surjective.

For a solution $u \in \mathrm{Sol}(\mathcal{E})$ we identify $M_u \simeq j_\infty(M) \subset J^\infty M$. Thus the characteristic variety is a bundle $\chi^{\mathcal{E}} \to M_u$, whose fiber at $x \in M_u$ is the projective variety

$$\chi^{\mathcal{E}}_x = \mathrm{Char}(\mathcal{E},u)_x = \{[\theta] \in \mathbb{P}(T_x^* M_u) : \sigma_F(\theta) = 0\}.$$

Quadratic characteristics

In coordinates to compute the characteristic variety one converts the symbol of linearization of F ("Fourier transform": $\partial_i \mapsto p_i$)

$$\sigma_F = \sum_{|\alpha|=\ell} \sigma_\alpha(u)\partial_\alpha \quad \text{to the polynomial} \quad \sigma_F(p) = \sum_{|\alpha|=\ell} \sigma_\alpha(u)p^\alpha$$

where $p = (p_1, \dots, p_d)$ is a coordinate on the fiber of T^*M_u and $p^\alpha = p_1^{i_1} \cdots p_d^{i_d}$ for a multi-index $\alpha = (i_1, \dots, i_d)$.

For second order PDEs the characteristic variety is a field of quadrics. We will assume it is nondegenerate and hyperbolic, i.e. $\det(\sigma_{ij}(u)) \neq 0$ and $\mathrm{Char}(\mathcal{E}^{\mathbb{C}}) = \mathrm{Char}(\mathcal{E})^{\mathbb{C}}$, respectively.

The nondegeneracy of σ_F implies that is inverse

$$g_F = \sum_{ij} g_{ij}(u)dx^i dx^j, \qquad (g_{ij}(u)) = (\sigma_{ij}(u))^{-1},$$

defines a symmetric bilinear form on T_xM_u. The canonical conformal structure $c_F = [g_F]$ on solutions of $\mathcal{E}$ is a base for geometric approach to integrability.

Systems and multiple characteristics

For the system we have a matrix representation of σ_F and the characteristic condition is its non-maximal rank.

Example: 3D

The Manakov-Santini system $\mathcal{E}$

$$P(u) = -u_x^2,\ P(v) = 0; \quad P = \partial_x\partial_t - \partial_y^2 + (u - v_y)\partial_x^2 + v_x\partial_x\partial_y,$$

has $\sigma_F(p) = \begin{pmatrix} \sigma_P(p) & 0 \\ 0 & \sigma_P(p) \end{pmatrix}$, where

$$\sigma_P(p) = p_x p_t - p_y^2 + (u - v_y)p_x^2 + v_x p_x p_y$$

and so the characteristic variety $\sigma_P(p) = 0$ (of multiplicity 2) is a nondegenerate quadric with the conformal structure:

$$g = -(dy - v_x dt)^2 + 4(dx - (u - v_y)dt)dt.$$

Example: 4D

The DFK master-equation $\mathcal{E}$ for SD

$$\partial_x Q(u) - \partial_y Q(v) = 0,$$
$$(\partial_w - u_y\partial_x + v_y\partial_y)Q(v) + (\partial_z + u_x\partial_x - v_x\partial_y)Q(u) = 0;$$
$$Q = \partial_x\partial_w + \partial_y\partial_z - u_y\partial_x^2 + (u_x + v_y)\partial_x\partial_y - v_x\partial_y^2,$$

has $\sigma_Q(p) = p_x p_w + p_y p_z - u_y p_x^2 + (u_x + v_y)p_x p_y - v_x p_y^2$ and

$$\sigma_F(p) = \begin{pmatrix} p_x\sigma_Q(p) & -p_y\sigma_Q(p) \\ (p_z + u_x p_x - v_x p_y)\sigma_Q(p) & (p_w - u_y p_x + v_y p_y)\sigma_Q(p) \end{pmatrix}$$

so that the characteristic variety $\sigma_Q(p) = 0$ (of multiplicity 3) is a nondegenerate quadric with the conformal structure on every solution of $\mathcal{E}$ given as follows:

$$g = dx\,dw + dy\,dz + u_y dw^2 - (u_x + v_y)\,dz\,dw + v_x dz^2.$$

Dispersionless Lax pairs (dLp)

Definition

A dispersionless pair of order N is a bundle $\hat{\pi}\colon \hat{M}_u \to M_u$ (correspondence space), whose fibres are connected curves, together with a rank 2 distribution $\hat{\Pi} \subseteq T\hat{M}_u$ such that:

- $\forall\ \hat{\boldsymbol{x}} \in \hat{M}_u$, $\hat{\Pi}(\hat{\boldsymbol{x}})$ depends only on $j^N_{\boldsymbol{x}} u$;
- $\hat{\Pi}$ is transversal to the fibres of $\hat{\pi}$.

Thus $\Pi(\hat{\boldsymbol{x}}) := (d\hat{\pi})_{\hat{\boldsymbol{x}}}(\hat{\Pi}) \subset T_{\boldsymbol{x}}M$ is a 2-plane congruence.
A spectral parameter is a local fibre coordinate λ on $\hat{\pi}$.

- $\hat{\Pi} \sim \hat{\Pi}'$ if $\forall\ u \in \mathrm{Sol}(\mathcal{E})$: $\hat{\Pi} = \hat{\Pi}'$ on $\hat{M}_u$;
- $\hat{\Pi}$ is a dispersionless Lax pair for $\mathcal{E}$ if for any $\hat{\Pi}' \sim \hat{\Pi}$, the Frobenius integrability condition $[\Gamma(\hat{\Pi}'), \Gamma(\hat{\Pi}')] \subseteq \Gamma(\hat{\Pi}')$ is a nontrivial differential corollary of $\mathcal{E}$.

First crucial ingredient: characteristic condition

Claim

A dLp $\hat{\Pi}$ is characteristic for $\mathcal{E}$, i.e. $\forall\ u \in \mathrm{Sol}(\mathcal{E})$, $\hat{\boldsymbol{x}} \in \hat{M}_u$ and $\theta \in \mathrm{Ann}(\Pi(\hat{\boldsymbol{x}})) \subseteq T^*_{\boldsymbol{x}} M_u$ we have $\sigma_F(\theta) = 0 \Leftrightarrow [\theta] \in \mathrm{Char}(\mathcal{E})$.

This means that for each solution u and $\hat{\boldsymbol{x}} \in \hat{M}_u$, $\Pi(\hat{\boldsymbol{x}})$ is a coisotropic 2-plane for the conformal structure c_F. Such 2-planes can only exist for $2 \leq d \leq 4$: for $d = 2$ the condition is vacuous; for $d = 3$ the coisotropic 2-planes at each point $\boldsymbol{x}$ form a rational conic $\mathbb{P}^1$; for $d = 4$ the form a disjoint union of two rational curves $2 \times \mathbb{P}^1$, the so-called α-planes and β-planes.

The passage from a 2-plane congruence $\Pi = \langle X, Y \rangle$ to a dLp can be understood as a lift, with respect to the projection $\hat{\pi}$:

$$\hat{X} = X + m\,\partial_\lambda, \quad \hat{Y} = Y + n\,\partial_\lambda.$$

The resulting rank 2 distribution $\hat{\Pi} = \langle \hat{X}, \hat{Y} \rangle$ is integrable mod $\mathcal{E}$ (on-shell), but not identically (off-shell).

Second crucial ingredient: projective condition

Definition

The dispersionless pair $\hat{\Pi} \subseteq T\hat{M}_u$ is normal if the derived distribution $\partial\hat{\Pi} = [\hat{\Pi}, \hat{\Pi}]$ is tangent to the $\hat{\pi}$-fibres for general u (off-shell), i.e. $\forall\ \hat{\boldsymbol{x}} = (\boldsymbol{x}, \lambda) \in \hat{M}_u$: $\hat{\pi}_*(\partial\hat{\Pi}(\hat{\boldsymbol{x}})) = \Pi(\boldsymbol{x}, \lambda)$.

Thus in this case the only integrability condition is the vanishing mod $\mathcal{E}$ of the vertical direction of the commutator $[\hat{X}, \hat{Y}] \bmod \hat{\Pi}$.

Claim

Let $d = 3$ and let Π be a nondegenerate quadratic 2-plane congruence on a $\mathbb{P}^1$-bundle $\hat{M} \to M$. Then Weyl connections parametrize normal lifts $\hat{\Pi}$ of Π such that the ∂_λ coefficient of a λ-independent vector field V on M_u is quadratic in λ for some choice of the spectral parameter λ (the projective property).

Note that for $d = 4$ no additional ingredient (connection) is required for the lift.

Theorem (D. Calderbank & BK 2016-2018)

Let $\mathcal{E} : F = 0$ be a nondegenerate determined PDE system in 3D or 4D whose characteristic variety is a quadric. Let c_F be the corresponding conformal structure. Then the integrability of $\mathcal{E}$ by a nondegenerate dispersionless Lax pair is equivalent to

3D: *the Einstein–Weyl property for c_F on any solution of the PDE;*

4D: *the self-duality property for c_F on any solution of the PDE.*

Proof: Given a dLp use its characteristic property to construct the correspondence space $\hat{M}_u$: $\hat{\Pi}$ is c_F-coisotropic on-shell, extend to a Zariski dense set of u off-shell. $\hat{\Pi}$ yields a 2-foliation on-shell, projects to $(d-1)$-parametric null totally geodesic foliation of M_u. By Cartan $(d=3)$ and Penrose $(d=4)$ this is equiv to EW/SD.

Conversely, let $\forall\, u \in \mathrm{Sol}(\mathcal{E})$ the structure c_F is EW or SD. Let $\hat{\pi} : \hat{M}_u \to M_u$ be the bundle of null 2-planes/α-planes. The normal lifts with projective property are bijective with Weyl connections for $d=3$, while for $d=4$ the normal lift is unique. This lifts the 2-plane congruence Π to a dLp $\hat{\Pi}$, unique up to equivalence. □

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Weyl potential $\rightsquigarrow$ dLp

Let $\mathcal{E}$ be a PDE with quadratic characteristic variety and such that its conformal structure c_F has EW property, with Weyl covector $\omega = \omega_i \theta^i$. Then $\mathcal{E}$ is integrable and the corresponding dispersionless Lax pair can be calculated explicitly (no integration).

Let us introduce the so-called null coframe $\theta^0, \theta^1, \theta^2$ (it depends on a finite jet of a solution $u \in \mathrm{Sol}(\mathcal{E})$) such that

$$g_F = 4\theta^0\theta^2 - (\theta^1)^2.$$

Let V_0, V_1, V_2 be the dual frame, and let c_{ij}^k be the structure functions defined by commutator expansions $[V_i, V_j] = c_{ij}^k V_k$. The Lax pair is given by vector fields

$$\hat{X} = V_0 + \lambda V_1 + m\partial_\lambda, \quad \hat{Y} = V_1 + \lambda V_2 + n\partial_\lambda,$$

where

$$m = (\tfrac{1}{2}c_{12}^1 - \tfrac{1}{4}\omega_2)\lambda^3 + (\tfrac{1}{2}c_{02}^1 - c_{12}^2 - \tfrac{1}{2}\omega_1)\lambda^2 + (\tfrac{1}{2}c_{01}^1 - c_{02}^2 - \tfrac{1}{4}\omega_0)\lambda - c_{01}^2,$$

$$n = -c_{12}^0\lambda^3 + (\tfrac{1}{2}c_{12}^1 - c_{02}^0 + \tfrac{1}{4}\omega_2)\lambda^2 + (\tfrac{1}{2}c_{02}^1 - c_{01}^0 + \tfrac{1}{2}\omega_1)\lambda + (\tfrac{1}{2}c_{01}^1 + \tfrac{1}{4}\omega_0)$$

General integrable systems in 3D

For Hirota type PDEs of the second order $F(u_{ij}) = 0$ in 3D integrability and Monge-Ampère property imply linearizability by a contact transformation. The general integrable equation is a modular form. The EW background structure is given by g_F and the following components of the Weyl covector

$$\omega_k = 2g_{kj}\mathcal{D}_{x^s}(g^{js}) + \mathcal{D}_{x^k}(\ln\det g_{ij}).$$

For general PDEs of second order $F(x^i, u, u_i, u_{ij}) = 0$ this formula is not applicable. Yet the EW structure can be determined.

Theorem (S.Berjawi, E.Ferapontov, BK, V.Novikov)

For nondegenerate non-Monge-Ampère equations of second order with EW property, the Weyl covector ω is algebraically determined.

Corollary

Under the above condition, the dispersionless Lax pair is algebraically determined by the equation.

General integrable systems in 4D

For Hirota type PDEs of the second order $F(u_{ij}) = 0$ in 4D integrability implies the Monge-Ampère property as proved by Ferapontov-BK-Novikov (2019). Such equations were investigated by Doubrov-Ferapontov (2010): classification over $\mathbb{C}$ consists of 6 versions of Plebanski heavenly equation (in fact, one is linear: ultra-wave PDE) obtained by deformations of

$$\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0, \quad \alpha + \beta + \gamma = 0.$$

For translation non-invariant PDEs we have:

Theorem (S.Berjawi, E.Ferapontov, BK, V.Novikov)

Every nondegenerate equation of second order with SD property must be of Monge-Ampère type. Freezing 1-jet of a solution we get one gets a PDE that is contact equivalent to one of Plebanski type heavenly equations.

References

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Homework: on Cartan's (2,3,5) and Engel distributions

- When $d=4$, then every 2-plane congruence Π on M_u has a unique normal lift. Check that generically, off-shell, $\Delta=\pi_*^{-1}(\Pi)\subset T\hat{M}_u$ is a nonholonomic rank 3 distribution with $[\Delta,\Delta]=T\hat{M}_u$, i.e., it has the growth vector (3,5) and, following Cartan, there is a unique rank 2 subbundle $\hat{\Pi}\subset\Delta$ with $[\hat{\Pi},\hat{\Pi}]=\Delta$. This rank 2 distribution has growth $(2,3,5)$ off-shell and is Frobenius integrable on-shell.

- When $d=3$, the normal lift of a 2-plane congruence Π is not unique. Instead, the rank 3 distribution $\Delta=\pi_*^{-1}(\Pi)\subset T\hat{M}_u$ has a unique Cauchy characteristic: a rank 1 subbundle $\mathcal{C}\subset\Delta$ with $[\mathcal{C},\Delta]=\Delta$. For a rank 2 subbundle $\hat{\Pi}\subset\Delta$ the normality condition $[\hat{\Pi},\hat{\Pi}]\subset\Delta$ implies that $\mathcal{C}\subset\hat{\Pi}$, but one generator of $\hat{\Pi}$ remains undetermined and is given by a choice of Weyl connection.