Dispersionless integrable systems (II)

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GRIEG seminar 2021

Systems of hydrodynamic type: $1+1$ D

Systems of hydrodynamic type $\mathcal{E}$ in 2D have the form

$$U_t = A(U)U_x \quad \Leftrightarrow u_t^i = a_j^i(u^1, \dots, u^n)u_x^j.$$

This first order evolutionary PDE is translationally invariant, and its characteristic variety is completely reducible over $\mathbb{C}$:

$$\mathrm{Char}(\mathcal{E}) = \{[p_t : p_x] \,|\, \det\bigl(A - \tfrac{p_t}{p_x}\mathbb{1}\bigr) = 0\}.$$

We consider systems that by a change of dependent variables $U \mapsto V(U)$ transform into a diagonal form in Riemann invariants

$$r_t^i = \lambda^i(\boldsymbol{r})r_x^i, \quad \boldsymbol{r} = (r^1, \dots, r^n) \qquad (\dagger)$$

(no summation by i). Then $\mathrm{Char}(\mathcal{E}) = \cup_{i=1}^n[\lambda^i : 1] \subset \mathbb{P}^1$.

The criterion of diagonalizability of the endomorphism field on the space of dependent vairables $\mathbb{R}^n(U)$ is the vanishing of the Haantjes tenor $0 = H_A \in \Lambda^2\mathbb{R}_n \otimes \mathbb{R}^n$, where

$$N_A(v,w) = [Av, Aw] - A[v, Aw] - A[Av, w] + A^2[v, w],$$

$$H_A(v,w) = N_A(Av, Aw) - AN_A(v, Aw) - AN_A(Av, w) + A^2N_A(v, w).$$

Gradient catastrophes

Contrary to dispersive equations, dispersionless PDEs often exhibit blow-up of solutions in finite time or gradient blow-up/singularities.

Example: Hopf (or inviscid Burgers') equation $r_t = r\, r_x$ can be rewritten as $dr \wedge (dx + r\, dt) = 0$, so that $dr = 0 \Leftrightarrow dx + r\, dt = 0$. This leads to multi-valued solutions given implicitly $x + rt = f(r)$. (See Maple for plots.)

Hugoniot-Rankine conditions use conservation laws to produce single-valued shock wave solutions. Dispersive perturbations lead to regularization of shocks.

Linearly degenerate systems are characterised by the condition $\partial_i \lambda_i = 0$, where $\partial_i = \partial_{r^i}$, for $i = 1, \dots, n$. Linear degeneracy prevents breakdown of smooth initial data $r^i(x, 0)$.

2D: Semi-Hamiltonian property

Diagonal system (†) is called semi-Hamiltonian if its characteristic speeds $\lambda^i(\boldsymbol{r})$ satisfy the relations (again $\partial_i = \partial_{r^i}$)

$$\partial_i\Bigl(\frac{\partial_j\lambda^k}{\lambda^j-\lambda^k}\Bigr)=\partial_j\Bigl(\frac{\partial_i\lambda^k}{\lambda^i-\lambda^k}\Bigr)\quad \forall i\neq j\neq k\neq i. \qquad (\ddagger)$$

For instance, $\forall\ c_i = \mathrm{const}$ and function ϕ of 1 argument the diagonal system with $\lambda^i = r^i + \phi(c_i r^i)$ is semi-Hamiltonian.

Relation to integrablity and Hamiltonian property.

Novikov conjecture: PDE system $\mathcal{E}$ is Hamiltonian and diagonalizable $\Rightarrow$ integrable.

Tsarev: Hamiltonian and diagonalizable $\Rightarrow$ semi-Hamiltonian with ∞ hydrodynamic conservation laws and symmetries.

Ferapontov: Reverse statement is wrong, but true for non-local Hamiltonians expressed as ∞ tail pseudo-differential operator.

Symmetries and conservation laws

Hydrodynamic symmetry corresponds to commuting flow

$$r^i_t = \lambda^i(\boldsymbol{r}) r^i_x, \quad r^i_\tau = \mu^i(\boldsymbol{r}) r^i_x \qquad (\triangle)$$

meaning compatibility $r^i_{t\tau} = r^i_{\tau t}$ due to $(\triangle)$, i.e. equation $\mathcal{E}$ together with its symmetry. This is equivalent to the relations

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \quad \forall i \neq j.$$

Given $\lambda^i(\boldsymbol{r})$ this is an overdetermined (for $n > 2$) linear PDE system on $\mu^i(\boldsymbol{r})$. It is compatible iff $\lambda^i(\boldsymbol{r})$ satisfy the semi-Hamiltonian property (‡). The travelling wave reductions correspond to $\lambda^i/\mu^i = \mathrm{const}$: $r^i = r^i(x, t + c\tau)$.

Thus system (†) is semi-Hamiltonian iff it possesses infinitely many symmetries parametrized by n functions of 1 variable.

Symmetries and conservation laws

Conservation laws $\omega = g\,dt + h\,dx$ with density $h(\boldsymbol{r})$ and flux $g(\boldsymbol{r})$ satisfy $d\omega = 0$ by virtue of (†) iff the following relations hold

$$\partial_i g = \lambda^i \partial_i h \qquad (\nabla)$$

(no summation), which by elimination of g (compatibility conditions) yields

$$\partial_i \partial_j h = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i h + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j h \quad \forall i \neq j.$$

Involutivity of this overdetermined PDE system is equivalent to the semi-Hamiltonian property (‡).

Thus system (†) is semi-Hamiltonian iff it possesses infinitely many conservation laws parametrized by n functions of 1 variable.

Hydrodynamic systems: Exact solutions

Given a hydrodynamic symmetry (△) the following implicit relation

$$x + \lambda^i(\boldsymbol{r})t = \mu^i(\boldsymbol{r}), \quad 1 \le i \le n$$

provides a solution $r^i = r^i(t, x)$, $i = 1, \dots, n$, to the PDE $\mathcal{E}$ given by (†). This is the generalized hodograph formula of Tsarev.

Since for semi-Hamiltonian systems, commuting flows μ^i depend on n arbitrary functions of 1 variable, the generalised hodograph formula provides a generic solution of system $\mathcal{E}$.

Similarly given a conservation law (∇) the nonlocal first integrals $f = \int \omega \Leftrightarrow \omega = df$ provide solutions to (†).

Note that if $\partial_i \lambda^j$ is symmetric in i, j (for instance when $N_A = 0$, which is a stronger condition than $H_A = 0$) then the equation for symmetries implies the equation for conservation laws (like in Noether theorem) with $\mu^i = \partial_i h$.

3D: hydrodynamic reductions

The method of hydrodynamic reductions in 3D applies to quasilinear differential equations $\mathcal{E}$ of the form

$$A(U)U_x + B(U)U_y + C(U)U_t = 0,$$

where (not necessary square) matrices A, B, C are such that the general solution of $\mathcal{E}$ depends on $m \geq 2$ functions of 2 variables. The method consists of reductions of the form

$$U = U(r^1, \dots, r^n)$$

and the phases satisfy a pair of commuting equations

$$r^i_t = \lambda^i(\boldsymbol{r})r^i_x, \quad r^i_y = \mu^i(\boldsymbol{r})r^i_x.$$

The commutativity condition of this system are precisely the same as for hydrodynamic symmetries of $1+1$ dimensional systems.

Definition

System $\mathcal{E}$ is integrable in hydrodynamic sense if $\forall n\ \exists\ n$-component reductions are parametrized by n arbitrary functions of 1 argument.

Multi-phase solutions: 3 implies ∞

One-phase solutions $U = U(r)$, $r = r(x, y, t)$, satisfy

$$r_t = \lambda(r) r_x, \quad r_y = \mu(r) r_x.$$

No obstructions. Such solutions are constant along one-parameter family of planes

$$x + \lambda(r)t + \mu(r)y + \nu(r) = 0$$

Two-phase solutions $U = U(r^1, r^2)$, $r^i = r^i(x, y, t)$, satisfy

$$r^i_t = \lambda^i(\boldsymbol{r}) r_x, \quad r^i_y = \mu^i(\boldsymbol{r}) r_x \quad (i = 1, 2).$$

Again, no obstructions. Such solutions are constant along two-parameter family of lines

$$x + \lambda^1(\boldsymbol{r})t + \mu^1(\boldsymbol{r})y + \nu^1(\boldsymbol{r}) = 0, \; x + \lambda^2(\boldsymbol{r})t + \mu^2(\boldsymbol{r})y + \nu^2(\boldsymbol{r}) = 0.$$

Three-phase solutions may not exist in general. In fact they impact integrability: the existence of general 3-phase solutions implies the existence of n-phase solutions for any $n \geq 3$.

Example of dKP

The dKP equation $(u_t - uu_x)_x = u_{yy}$ can be rewritten in the first-order (hydrodynamic) form

$$u_t - uu_x = v_y, \qquad u_y = v_x.$$

Then n-phase solutions $u = u(r^1, \dots, r^n)$, $v = v(r^1, \dots, r^n)$

$$r^i_t = \lambda^i(\boldsymbol{r}) r^i_x, \qquad r^i_y = \mu^i(\boldsymbol{r}) r^i_x.$$

satisfy the following relations eliminating $\lambda^i(\boldsymbol{r}), v(\boldsymbol{r})$

$$\lambda^i = u + (\mu^i)^2, \qquad \partial_i v = \mu^i \partial_i u,$$

together with the following constraints on $u(\boldsymbol{r}), \mu^i(\boldsymbol{r})$ for $i \neq j$:

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \qquad \partial_{ij} u = \frac{2\,\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}.$$

This Gibbons-Tsarev system is in involution, so its general local solution depends on ~~$2n$~~ n functions of 1 variable.

Higher dimensional hydrodynamic reductions

Similarly we can treat higher dimensional cases. For instance in 4D the first Plebansky heavenly equation

$$u_{tx}u_{yz} - u_{ty}u_{xz} = 1$$

can be rewritten in the first order quasi-linear form via the substitution $u_{tx} = a$, $u_{yz} = b$, $u_{ty} = c$, $u_{xz} = \frac{ab-1}{c}$:

$$a_y = c_x,\ a_z = \left(\frac{ab-1}{c}\right)_t,\ b_t = c_z,\ b_x = \left(\frac{ab-1}{c}\right)_y.$$

Expressing these in terms of Riemann invariants r^i subject to

$$r^i_t = \lambda^i(\boldsymbol{r})r^i_x, \quad r^i_y = \mu^i(\boldsymbol{r})r^i_x, \quad r^i_z = \eta^i(\boldsymbol{r})r^i_x,$$

we obtain another Gibbons-Tsarev system in involution, whence a general local solution depending on ~~$3n$~~ $2n$ functions of 1 variable.

Definition

System $\mathcal{E}$ with d independent variables is integrable in hydrodynamic sense if $\forall n\ \exists\ n$-component reductions are parametrized by $(d-2)n$ arbitrary functions of 1 argument.

Equivalence of approaches

There are several types of PDE systems that are subject to the method of hydrodynamic reductions. For instance, Hirotha type equations

$$F(\partial^2 u) = 0, \qquad \text{where } \partial^2 u = (u_{ij} : 1 \leq i \leq j \leq d)$$

or quasilinear second order PDE

$$\sum a^{ij}(\partial u) u_{ij} = 0, \qquad \text{where } \partial u = (u_i : 1 \leq i \leq d).$$

Theorem (E. Ferapontov & BK 2014)

Let $\mathcal{E}$ be a nondegenerate determined PDE system in 3D or 4D of one of the above types. Then its integrability by the method of hydrodynamic reductions is equivalent to the existence of a dispersionless Lax pair (dLp). Moreover, the PDE is a reduction of an integrable background geometry: EW in 3D and SD in 4D.

In 3D the Lax pair can be also taken as dZp in 3D.

A Weyl structure on M^3 is the pair $([g],\mathbb{D})$: a conformal structure and a linear connection preserving it. This condition writes so:

$$\mathbb{D}g=\omega\otimes g.$$

A choice of via 1-form ω is equivalent to a choice of $\mathbb{D}$.

For the general linear connection $\mathbb{D}$, its Ricci tensor $\mathrm{Ric}_{\mathbb{D}}$ needs not be symmetric: $\mathrm{Ric}_{\mathbb{D}}^{\mathrm{alt}}\sim d\omega$. The Einstein-Weyl equation is

$$\mathrm{Ric}_{\mathbb{D}}^{\mathrm{sym}}=\Lambda\, g\qquad\text{for some } \Lambda\in C^\infty(M).$$

The pair $([g],\mathbb{D})$ is an Einstein-Weyl structure if it satisfies the above 5 second order PDEs on 5 entries of the conformal structure and 3 of the covector.

For $\omega=0$ the connection $\mathbb{D}$ is Levi-Civita, and the above is just the Einstein equation. EW are generalize Einstein structures.

In 4D the Weyl curvature W is the fundamental invariant of a conformal structure $[g]$. The Hodge operator acts on the space $(S^2\Lambda^2TM)_0$ of Weyl tensors and it is an involution for Riemannian or neutral signature, whence the split $W = W_+ + W_-$ into self-dual and anti-self-dual parts. The structure $[g]$ is self-dual if $W_- = 0$. These are 5 second order PDEs on the 9 entries of $[g]$:

$$*W_g = W_g.$$

Both EW and SD (or ASD) equations are Lax-integrable, as well as some of their reductions, e.g. anti-self dual Einstein (=heavenly) equations. Several other PDEs have been obtained as (~~symmetry~~) reductions of the two equations, allowing to think of them as master-equations in 3D and 4D respectively.

Explicit form of EW system

According to Hitchin, the system of EW equations is integrable (via twistor theory). We will write its PDEs in a proper gauge.

Theorem (M. Dunajski, E. Ferapontov & BK 2015)

Any Lorentzian Einstein-Weyl structure is locally of the form

$$g = -(dy - v_x dt)^2 + 4(dx - (u - v_y)dt)dt,$$
$$\omega = -v_{xx}dy + (4u_x - 2v_{xy} + v_x v_{xx})dt,$$

where the functions u, v *on* M^3 *satisfy*

$$P(u) = -u_x^2,\ P(v) = 0; \quad P = \partial_x\partial_t - \partial_y^2 + (u - v_y)\partial_x^2 + v_x\partial_x\partial_y.$$

The above coupled second-order PDE system, known as the Manakov-Santini (MS) system, has the Lax pair

$$L_1 = \partial_y - (\lambda + v_x)\partial_x - u_x\partial_\lambda,$$
$$L_2 = \partial_t - (\lambda^2 + v_x\lambda - u + v_y)\partial_x - (u_x\lambda + u_y)\partial_\lambda.$$

Explicit form of SD/ASD equations

According to Penrose, the system of (A)SD equations is integrable (via twistor theory). We will write its PDEs in a proper gauge.

Theorem (M. Dunajski, E. Ferapontov & BK 2015)

Any ASD conformal structure of signature (2,2) has local form

$$g = dx\,dw + dy\,dz + u_y dw^2 - (u_x + v_y)\,dz\,dw + v_x dz^2,$$

where the functions u, v *on* M^4 *satisfy*

$$\partial_x Q(u) - \partial_y Q(v) = 0,$$
$$(\partial_w - u_y\partial_x + v_y\partial_y)Q(v) + (\partial_z + u_x\partial_x - v_x\partial_y)Q(u) = 0,$$
$$Q = \partial_x\partial_w + \partial_y\partial_z - u_y\partial_x^2 + (u_x + v_y)\partial_x\partial_y - v_x\partial_y^2.$$

The above coupled third-order PDE system (DFK) has the Lax pair

$$L_1 = \partial_w - u_y\partial_x + (\lambda + v_y)\partial_y + Q(u)\partial_\lambda,$$
$$L_2 = \partial_z + (\lambda + u_x)\partial_x - v_x\partial_y - Q(v)\partial_\lambda.$$

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Homework: SD/EW

- $\mathbb{C}P^2$ has SD structure, and $\overline{\mathbb{C}P^2}$ has ASD structure. What about $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$?
- Quotient $\mathbb{C}P^2 = SU(3)/U(2)$ equipped with FS metric by the Killing field from $SO(2)$. What's the corresponding EW str?
- Consider the round metric on S^3. How many Weyl covectors ω satisfying EW does it possess mod isometry group $SO(3)$?