Dispersionless integrable systems (I)

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Dynamical systems: $d = 1$

Let me first recall integrability of ODEs. Locally on M^n to integrate the differential equation $\dot{x} = v(x)$ one needs $n-1$ first integrals $I_1, \dots, I_{n-1}$, i.e. $L_v(I_i) = 0$ (the functions I_i are assumed functionally independent $dI_1 \wedge \dots \wedge dI_{n-1} \not\equiv 0$). The trajectories are given by $\{I_i = c_i\}_{i=1}^{n-1}$. When M is closed this is too restrictive: the system would be resonant, i.e. have all trajectories periodic.

If the system is Hamiltonian $\dot{x} = X_H = \omega^{-1} dH$ for a symplectic form ω on W^{2n}, then integrability by Liouville requires $n = \frac{1}{2} \dim W$ first integrals, if they are in involution. Since $H = E$ is a trivial integral, this means the existence of $I_1, \dots, I_{n-1}$ such that $\{H, I_i\} = 0 = \{I_i, I_j\}$, where $\{I, J\} = \pi(dI, dJ)$ for $\pi = \omega^{-1} \in \Gamma(\Lambda^2 TW)$; again we assume functional independence $dI_1 \wedge \dots \wedge dI_{n-1}|_{H=E} \not\equiv 0$. Then the motion is quasi-periodic.

Note that $\{H, I\} = X_H(I) = -X_I(H)$, so X_I is a symmetry of H iff I is the first integral (Noether theorem).

2D: Integrability in $1 + 1$ dimensions

Different approaches to integrability:

- Lax pairs in differential operators
- Infinity of higher symmetries
- Infinity of conservation laws
- Bi-Hamiltonian structure, recursion operator
- Exact (soliton) solutions
- Existence of a Bäcklund transformation
- Existence of a Wahlquist-Estabrook prolongation structure

We will demonstrate this on the example of the Korteweg de Vries equation (KdV)

$$u_t = 6uu_x - u_{xxx} \qquad (\dagger)$$

note that scaling $(t, x) \mapsto (at, bx)$ changes the factors $(6, -1)$ at the rhs to any desired pair of nonzero numbers.

KdV: Lax pair

A Lax pair for a nonlinear PDE is an overdetermined linear system (depending on auxiliary spectral parameter λ) whose compatibility condition is the given PDE. The KdV (†) has Lax pair:

$$v_{xx} = (u - \lambda)v,$$
$$v_t = 2(u + 2\lambda)v_x - u_x v.$$

Geometrically the KdV equation is a submanifold $\mathcal{E} \subset J^\infty(\mathbb{R}^2_{t,x}\mathbb{R}^1_u)$ equipped with the Cartan distribution $\mathscr{C} = \langle \mathcal{D}_t, \mathcal{D}_x \rangle$, where

$$\mathcal{D}_t = \partial_t + \sum u_{\sigma,t}\partial_{u_\sigma}|_{\mathcal{E}}, \quad \mathcal{D}_x = \partial_x + \sum u_{\sigma,x}\partial_{u_\sigma}|_{\mathcal{E}}.$$

Its Lax pair gives the submanifold-equation $\tilde{\mathcal{E}} \subset J^\infty(\mathbb{R}^2_{t,x}\mathbb{R}^2_{u,v})$ equipped with the Cartan distribution $\tilde{\mathscr{C}} = \langle \tilde{\mathcal{D}}_t, \tilde{\mathcal{D}}_x \rangle$. The submersion $\pi : (\tilde{\mathcal{E}}, \tilde{\mathscr{C}}) \to (\mathcal{E}, \mathscr{C})$ is called differential covering (the inverse: integrable extension); it is a sign of integrability.

KdV: Higher symmetries (commuting flows)

A symmetry of equation $\mathcal{E} \subset J^\infty$ is a contact vector field X_φ s.t. $X_\varphi^{(\infty)} \in \Gamma(T\mathcal{E})$. If $\varphi \in C^\infty(J^1)$ then X is a classical symmetry (point if φ is affine in 1-jet; contact if nonlinear - possible only for scalar PDEs), otherwise it is called a higher symmetry.

With the notations $u_0 = u$, $u_1 = u_x$, $u_2 = u_{xx}$, etc, the (first) higher symmetries of (†) are:

$$S_0 = u_1,\ S_1 = 6u_0u_1 - u_3,\ S_2 = 30u_0^2u_1 - 20u_1u_2 - 10u_0u_3 + u_5,$$
$$S_3 = 140u_0u_1(u_0^2 - 2u_2) - 70u_1^3 + 70(u_2 - u_0^2)u_3 + 42u_1u_4 + 14u_0u_5 - u_7, \dots$$

In fact, there is one new higher symmetry in every odd order. This means the flows $u_{t_0} = S_0, u_{t_1} = S_1, u_{t_2} = S_2, \dots$ commute.

If $\mathcal{E} = \{F = 0\}$ and $\ell_F = \sum(\partial_{u_\sigma}F)\mathcal{D}_\sigma$ is the linerization operator, then the equation for symmetries $\mathrm{sym}(\mathcal{E}) = \langle X_\varphi \rangle$ is

$$\ell_F(\varphi)|_{\mathcal{E}} = 0 \ \Leftrightarrow \{F, \varphi\} = 0 \mod \mathcal{E},\ \text{where } \{F, G\} = \ell_F(G) - \ell_G(F).$$

KdV: conservation laws

By the Noether theorem every symmetry gives a conservation law, which is such a form $\omega \in \Omega^{d-1}_{\text{hor}}(\mathcal{E})$ that $\hat{d}\omega = 0 \mod \mathcal{E}$. For $d = 1$ this is a first integral, for $d = 2$ for an evolutionary PDE we have:

$$h(x, t, u, u_x, \dots)_t + f(x, t, u, u_x, \dots)_x = 0.$$

Here $\omega = f dt - h dx$, h is called the conserved density, f - the flux. The conservation law is recovered from its density h, which in turn can be found from the generating function of a symmetry.

In the example of KdV (†) we have:

$$u_t + (u_{xx} - 3u^2)_x = 0,$$
$$(u^2)_t + (2uu_{xx} - u_x^2 - 4u^3)_x = 0,$$
$$(u^3 + \tfrac{1}{2}u_x^2)_t + (u_x u_{xxx} - \tfrac{1}{2}u_{xx}^2 + 3u^2 u_{xx} - 6uu_x^2 - \tfrac{9}{2}u^4)_x = 0,$$

giving the conserved quantities $\int u\,dx$, $\int u^2\,dx$, $\int (u^3 + \frac{1}{2}u_x^2)\,dx$ known as the mass (Casimir), momentum and energy.

KdV: Hamiltonian structures, recursion operator

The KdV equation possesses bi-Hamiltonian structure (in fact, infinitely many of compatible Hamiltonian structures)

$$u_t = J_1 \frac{\delta I_1}{\delta u} = J_2 \frac{\delta I_2}{\delta u},$$

where the Poisson bracket is $\{F, G\}_J = \int \frac{\delta F}{\delta u} J \frac{\delta G}{\delta u}$ and

$$J_1 = \mathcal{D}_x, \quad I_1 = \int_{-\infty}^{\infty} \left(u^3 + \tfrac{1}{2} u_x^2\right) dx;$$

$$J_2 = -\mathcal{D}_x^3 + 4u\mathcal{D}_x + 2u_x, \quad I_2 = \frac{1}{2} \int_{-\infty}^{\infty} u^2 \, dx.$$

The recursion operator R is defined as

$$R = J_2 J_1^{-1} = -\mathcal{D}_x^2 + 4u + 2u_x \mathcal{D}_x^{-1}.$$

With the help of this we can iterate the higher symmetries

$$S_0 \xrightarrow{R} S_1 \xrightarrow{R} S_2 \xrightarrow{R} S_3 \xrightarrow{R} S_4 \xrightarrow{R} \dots$$

KdV: Soliton solutions

The inverse scattering method allows to solve (†) via linear ODEs. In particular it gives special localized solutions that retain their size and shape all the time, even when they pass through each other.

They are among stationary points of higher symmetries.

The first soliton is obtain by symmetry reduction $u_t + 4k^2 u_x = 0$:

$$u(t,x) = -2k^2 \cosh(k(x-4k^2t))^{-2}.$$

More general, n-soliton solutions are given by the explicit formula

$$u_n(t,x) = -2\partial_x^2 \ln \det A(t,x),$$

where A is $n \times n$ matrix

$$A_{ij} = \delta_{ij} + \frac{c_i}{k_i + k_j} e^{-(k_i+k_j)x + 8k_i^3 t}.$$

Asymptotically as $t \to \pm\infty$ we have for some $q_i^{\pm}$:

$$u_n(t,x) \sim -2 \sum_{i=1}^{n} k_i^2 \cosh(k_i(x - 4k_i^2 t - q_i^{\pm}))^{-2}.$$

3D: integrability of KP equation

The Kadomtsev-Petviashvili equation (KP) is an extension of KdV:

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \qquad (\ddagger)$$

It can be written in evolutionary form as $u_t = 6uu_x - u_{xxx} - 3\mathcal{D}_x^{-1}u_{yy}$ and then one can associate Hamiltonian formalism: non-local for this form but local for (‡). There are infinitely many higher symmetries and conservation laws for KP, however they are non-local.

What is more important there is still a Lax pair:

$$v_t = -4v_{xxx} + 6uv_x + 3(u_x - \lambda + w)v,$$
$$v_y = v_{xx} - uv.$$

Compatiblity of this overdetermined system is equivalent to

$$u_t = 6uu_x - u_{xxx} - 3w_y,\ u_y = w_x.$$

Eliminating from this $w = \mathcal{D}_x^{-1}u_y$ yields (‡).

From KP to dKP: removing dispersion

Following Zakharov, consider the fast oscillation limit $\epsilon \to 0$ under

$$v = e^{\psi/\epsilon}, \quad \partial_t \mapsto \epsilon\partial_t, \quad \partial_x \mapsto \epsilon\partial_x, \quad \partial_y \mapsto \epsilon\partial_y.$$

This transforms the previous Lax pair (spectral parameter λ removed) into (we call such dispersionless Zakharov coverings dZp)

$$G_1 : \quad \psi_t = -4\psi_x^3 + 6u\psi_x + 3w,$$
$$G_2 : \quad \psi_y = \psi_x^2 - u.$$

The compatibility conditions for this system $\{G_1, G_2\} = 0 \bmod \langle G \rangle$ are $u_t = 6uu_x - 3w_y$, $u_y = w_x$, yielding under elimination of w:

$$u_{tx} = 6(uu_x)_x - 3u_{yy}.$$

This is the dispersionless limit of the Kadomtsev-Petviashvili equation called dKP.

From dZp to dLp

Let $\mathcal{E}$ be a PDE on a manifold M, u its solution. Denote by M_u the pair (M, u) realized as a section (submanifold) of $\mathcal{E} \subset J^\infty$.

A dispersionless Lax pair (dLp) is a rank 2 distribution on a projective line bundle $\hat{M}_u \to M_u$ for any $u \in \Gamma(J^\infty)$ such that the Frobenius integrability follows from (equivalent to) $\mathcal{E}$.

For dKP, with λ the fiber-coordinate (spectral parameter):

$$\hat{\Pi}^2 = \langle \partial_t + (12\lambda^2 - 6u)\partial_x + (6\lambda u_x + 3u_y)\partial_\lambda, \partial_y - 2\lambda\partial_x - u_x\partial_\lambda \rangle.$$

How to get this dLp from dZp? Consider the symplectic manifold $T^*M_u \simeq \mathbb{R}^6(t, x, y, \psi_t, \psi_x, \psi_y)$ equipped with

$$\Omega = dt \wedge d\psi_t + dx \wedge d\psi_x + dy \wedge d\psi_y.$$

Then dZp determines a codim$=2$ submanifold N. The restriction $\Omega|_N$ has rank 2 iff u satisfies dKP. In this case $\mathrm{Ker}(\Omega|_N)$ is a rank 2 distribution and, denoting $\lambda = \psi_x$, we get the above $\hat{\Pi}^2$.

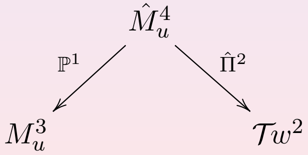
Geometric interpretation of dLp

How to get dZp from dLp? In general this is impossible. Yet one can get a linear covering as follows. Write the equation for integral surfaces: $L_v(\lambda - \lambda(\boldsymbol{x})) = 0 \ \forall v \in \Gamma(\hat{\Pi}^2)$, where $\boldsymbol{x} = (t, x, y)$:

$$\begin{aligned} \lambda_t &= (6u - 12\lambda^2)\lambda_x + 6\lambda u_x + 3u_y, \\ \lambda_y &= 2\lambda\lambda_x - u_x. \end{aligned} \qquad (\natural)$$

The compatibility of this system is equivalent to dKP.

The manifold $\hat{M}_u$ is called the correspondence space, the local leaf space of $\hat{\Pi}$ is the twistor space. Thus we arrive at double fibration



Solutions to $(\natural)$ are curves in $\mathcal{T}w^2$ while solutions to $L_v(\phi) = 0$ $\forall v \in \Gamma(\hat{\Pi}^2)$ are functions on this (mini-)twistor space.

Further signs of integrabiltiy

Instead of higher (local) symmetries for dispersionless systems we get integrable hierarchies (filtered by compatible systems of increasing size). For instance, we have the potential dKP hierarchy

$$u_{i,j+1} - u_{j,i+1} + \sum_{k=1}^{i} u_{i-k} u_{jk} - \sum_{k=1}^{j} u_{j-k} u_{ik} = 0$$

that is compatible for any $1 \leq i < j \leq n$ ($u_i = u_{x^i}$, etc). Here $u = u(x^1, x^2, x^3, x^4, \dots)$ and $x^1 = x$, $x^2 = y$, $x^3 = t$, $x^4 = z$, etc. These equations have Lax representation in vector fields

$$X_i = \partial_{x^{i+1}} - \lambda \partial_{x^i} - \sum_{k=1}^{i-1} u_{i-k} \partial_{x^k} + u_{1i} \partial_\lambda, \ i \geq 1.$$

The initial equations are

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

$$u_{xz} - u_x u_{xy} - u_y u_{xx} - u_{yt} = 0, \quad u_{yz} - u_y u_{xy} + u_x^2 u_{xx} - u_{tt} = 0,$$

the first being related to the dKP by a scaling and potentiation.

4D: dispersionless integrability

In the same way integrability is accessed for $d = 4$. Again a dLp is a rank 2 distribution on a projective line bundle $\hat{M}_u \to M_u$ whose Frobenius condition implies (equivalent to) the equation $\mathcal{E}$ on u. The twistor picture is as follows.

The manifold $\hat{M}_u$ is called the correspondence space, the local leaf space of $\hat{\Pi}$ is the twistor space. Thus we arrive at double fibration

$$M_u^4 \xleftarrow{\ \mathbb{P}^1\ } \hat{M}_u^5 \xrightarrow{\ \hat{\Pi}^2\ } \mathcal{T}w^3$$

Solutions to $L_v(\lambda - \lambda(\boldsymbol{x})) = 0\ \forall v \in \Gamma(\hat{\Pi}^2)$ are surfaces in $\mathcal{T}w^3$; solutions to $L_v(\phi) = 0\ \forall v \in \Gamma(\hat{\Pi}^2)$ are functions on this twistor space.

Ex: Plebanski heavenly equations

For the second Plebanski equation

$$u_{tx} + u_{yz} + u_{xx}u_{yy} - u_{xy}^2 = 0$$

the dLp is given by $\hat{\Pi}^2 = \langle \hat{T}, \hat{Z} \rangle$ with

$$\hat{T} = \partial_t + u_{yy}\partial_x - (u_{xy} - \lambda)\partial_y, \quad \hat{Z} = \partial_z - (u_{xy} + \lambda)\partial_x + u_{xx}\partial_y.$$

This yields the following contact covering

$$q_t = (u_{xy} - q)q_y - u_{yy}q_x,$$
$$q_z = (u_{xy} + q)q_x - u_{xx}q_y.$$

Compatibility of this system (commutativity of $\hat{T}, \hat{Z}$) is a consequence of the Plebanski equation.

References

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Homework: twistors

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\begin{array}{ccc}
 & D_3/P_{1,2} & \\
\swarrow^{\mathbb{P}^1} & & \searrow^{\mathbb{P}^2} \\
D_3/P_1 & & D_3/P_2
\end{array}
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