

GRIEG LECTURE 3

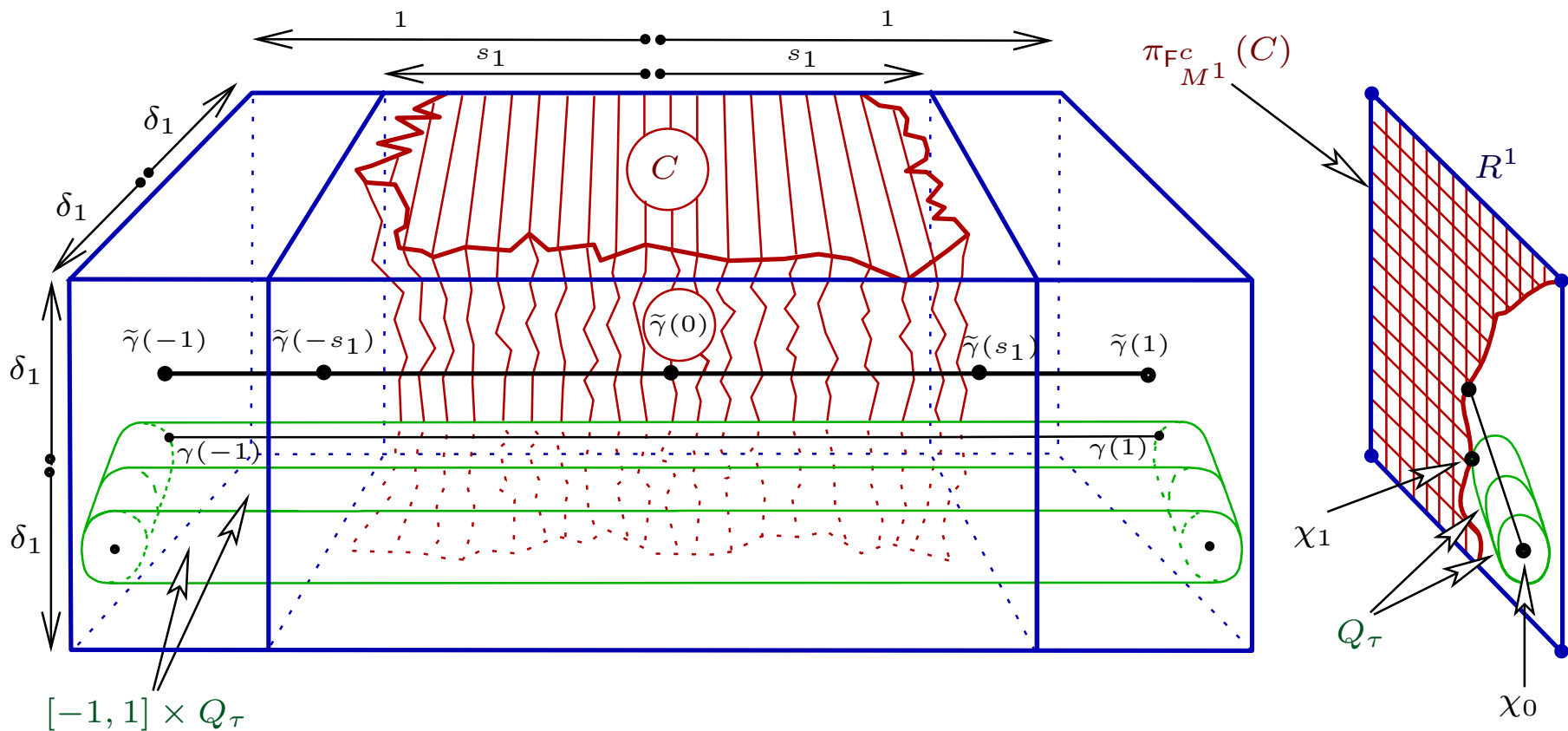
JOËL MERKER

www.imo.universite-paris-saclay.fr/~merker/

Département de Mathématiques d'Orsay

Université Paris-Saclay

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1. Affinely Homogeneous Parabolic Surfaces $S^2 \subset \mathbb{R}^3$

After translation, an affine transformation of \mathbb{R}^3 fixes the origin. Consider therefore a linear map $(x, y, u) \mapsto (r, s, v)$:

$$\begin{aligned} r &:= a_{1,1}x + a_{1,2}y + b_1u, \\ s &:= a_{2,1}x + a_{2,2}y + b_2u, \\ v &:= c_1x + c_2y + du, \end{aligned} \quad \text{with} \quad 0 \neq \begin{vmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & b_{2,2} & b_2 \\ c_1 & c_2 & d \end{vmatrix}.$$

Also, consider two local analytic surfaces passing through the origin, graphed as:

$$u = F(x, y) \quad (F(0,0) = 0) \quad \text{and} \quad v = G(r, s) \quad (0 = G(0,0)),$$

with convergent series:

$$F = \sum_{i+j \geq 1} F_{j,k} \frac{x^i}{i!} \frac{y^j}{j!} \quad \text{and} \quad G = \sum_{k+l \geq 1} G_{k,l} \frac{r^k}{k!} \frac{s^l}{l!}.$$

The linear map above sends the left surface $\{u = F\}$ to the right surface $\{v = G\}$ if and only if the *fundamental equation*:

$$(1.1) \quad 0 \equiv \text{eqFG}(x, y),$$

holds identically in $\mathbb{R}\{x, y\}$, where:

$$\begin{aligned} \text{eqFG} := & -c_1 x - c_2 y - d F(x, y) \\ & + G\left(a_{1,1}x + a_{1,2}y + b_1 F(x, y), a_{2,1}x + a_{2,2}y + b_2 F(x, y)\right), \end{aligned}$$

so that:

$$0 = \text{eqFG} = \sum_{i,j \in \mathbb{N}} \mathcal{C}_{i,j} \left(a_{\bullet,\bullet}, b_{\bullet}, c_{\bullet}, d_{\bullet}, F_{\bullet,\bullet}, G_{\bullet,\bullet} \right) x^i y^j.$$

The *core work* is to *compute* these (often complicated) coefficients $\mathcal{C}_{i,j} = 0$ and to *analyze* their vanishing.

For all $i, j \in \mathbb{N}$, the coefficient of $x^i y^j$ in eqFG can, in a standard way, be denoted as:

$$[x^i y^j] \text{eqFG} := \mathcal{C}_{i,j} = 0,$$

and we will constantly indicate the corresponding indices \mathbf{i}, \mathbf{j} over the equal sign as:

$$0 \stackrel{\mathbf{i}, \mathbf{j}}{=} \mathcal{C}_{i,j}.$$

We will proceed inductively, *order by order*, where:

$$\text{order} := i + j.$$

Two obvious affine transformations make horizontal the two tangent spaces:

$$u = \mathbf{0} + O_{x,y}(2) \quad \text{and} \quad v = \mathbf{0} + O_{r,s}(2),$$

where the two $\mathbf{0}$ should be interpreted as *normalizations* of order 1 terms.

Lemma 1.2. *Stabilization of order 1 terms holds if and only if $0 = c_1 = c_2$:*

$$\begin{bmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ c_1 & c_2 & d \end{bmatrix}^{\mathbf{0}} \rightsquigarrow \begin{bmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ \mathbf{0} & \mathbf{0} & d \end{bmatrix}^{\mathbf{1}}.$$

Proof. Read from (1.1):

$$0 \stackrel{\mathbf{1,0}}{=} -c_1,$$

$$0 \stackrel{\mathbf{0,1}}{=} -c_2. \quad \square$$

Next, pass to order 2. Possibly after rotation in the (x, y) -space and in the (r, s) -space, the constant Hessian rank 1 hypothesis — which is

affinely invariant — reads as:

$$F_{xx} \neq 0 \equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} \quad \begin{matrix} \text{known} \\ \iff \end{matrix} \quad G_{rr} \neq 0 \equiv \begin{vmatrix} G_{rr} & G_{rs} \\ G_{sr} & G_{ss} \end{vmatrix}.$$

Thus at order 2, we have to normalize two rank 1 basic quadratic forms:

$$F_{2,0} \frac{x^2}{2} + F_{1,1} xy + F_{0,2} \frac{y^2}{2} \quad \text{and} \quad G_{2,0} \frac{r^2}{2} + G_{1,1} rs + G_{0,2} \frac{s^2}{2}.$$

Proposition 1.3. *Two appropriate linear transformations in the (x, y) -space and in the (r, s) -space normalize:*

$$u = \frac{x^2}{2} + O_{x,y}(3) \quad \text{and} \quad v = \frac{r^2}{2} + O_{r,s}(3).$$

Furthermore, stabilization of order 2 terms holds if and only if $a_{1,2} = 0$ and $d = a_{1,1}^2$:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ \mathbf{0} & \mathbf{0} & d \end{bmatrix}^{\mathbf{1}} \quad \rightsquigarrow \quad \begin{bmatrix} a_{1,1} & \mathbf{0} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{2}}.$$

Proof. The first assertion is known. Then the second follows by computing the three equations $\overset{\mathbf{2,0}}{=}$, $\overset{\mathbf{1,1}}{=}$, $\overset{\mathbf{0,2}}{=}$ from (1.1). □

All this is in fact proved in arxiv.org/abs/2202.03030/ for constant Hessian rank 1 hypersurfaces $H^n \subset \mathbb{R}^{n+1}$, in any dimension $n \geq 1$.

Next, let order 3 monomials appear:

$$u = \frac{x^2}{2} + F_{3,0} \frac{x^3}{6} + F_{2,1} \frac{x^2 y}{2} + F_{1,2} \frac{x y^2}{2} + F_{0,3} \frac{y^3}{6} + O_{x,y}(4),$$

$$v = \frac{r^2}{2} + G_{3,0} \frac{r^3}{6} + G_{2,1} \frac{r^2 s}{2} + G_{1,2} \frac{r s^2}{2} + G_{0,3} \frac{s^3}{6} + O_{r,s}(4).$$

Then Hessian rank 1 implies (exercise) $0 = F_{1,2} = F_{0,3}$ and $G_{1,2} = G_{0,3} = 0$:

$$\begin{array}{l} u = \frac{x^2}{2} + F_{3,0} \frac{x^3}{6} + F_{2,1} \frac{x^2 y}{2} + O_{x,y}(4) \\ \xrightarrow{\text{Equivalence}} v = \frac{r^2}{2} + G_{3,0} \frac{r^3}{6} + G_{2,1} \frac{r^2 s}{2} + O_{r,s}(4). \end{array}$$

Now starts the real work. The fundamental equation gives:

$$0 \stackrel{\mathbf{3,0}}{=} -a_{1,1}^2 F_{3,0} + a_{1,1}^3 G_{3,0} + 3 a_{1,1}^2 a_{2,1} G_{2,1} + 3 a_{1,1} \boxed{b_1},$$

$$0 \stackrel{\mathbf{2,1}}{=} -a_{1,1}^2 F_{2,1} + a_{1,1}^2 a_{2,2} G_{2,1}.$$

while $\stackrel{\mathbf{1,2}}{=}$ and $\stackrel{\mathbf{0,3}}{=}$ bring nothing for they both reduce to $0 = 0$.

Observe that since the stability group at order 2 is a subgroup of $GL(3, \mathbb{R})$:

$$0 \neq \begin{vmatrix} a_{1,1} & 0 & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ 0 & 0 & a_{1,1}^2 \end{vmatrix} = a_{1,1} a_{2,2} a_{1,1}^2,$$

we have $a_{1,1} \neq 0$, and therefore, the boxed free group parameter $\boxed{b_1}$ can be used to normalize:

$$G_{3,0} := \mathbf{0},$$

just by assigning:

$$b_1 := \frac{1}{3} a_{1,1} F_{3,0} - \mathbf{0} - a_{1,1} a_{2,1} G_{2,1},$$

replacing of course $G_{2,1} = \frac{1}{a_{2,2}} F_{2,1}$ from $\mathbf{2,1}$.

Once $G_{3,0} = 0$ is so normalized, we restart from the surface on the right $\{v = G\}$, we place it on the left, we change notation $(r, s, v) \mapsto (x, y, u)$, $G \mapsto F$, we rename it $\{u = F\}$ thus with $F_{3,0} = 0$, we take another affine equivalence to another surface $\{v = G\}$ on the right, and we again normalize similarly $G_{3,0} = 0$.

Thus without any further work, we can assume $F_{3,0} = 0 = G_{3,0}$, *simultaneously*.

Generally, once a normalization has been made on the right, always, it can also be made *exactly the same* on the left.

Principle 1.4. *At any order, every performed normalization will always be instantly achieved on both hypersurfaces $\{u = F\}$ and $\{v = G\}$. \square*

Thus:

$$u = \frac{x^2}{2} + F_{2,1} \frac{x^2 y}{2} + O_{x,y}(4) \xrightarrow{\text{Equivalence}} v = \frac{r^2}{2} + G_{2,1} \frac{r^2 s}{2} + O_{r,s}(4).$$

Next, since $a_{1,1} \neq 0 \neq a_{2,2}$, the remaining equation $\stackrel{2,1}{=}$, namely:

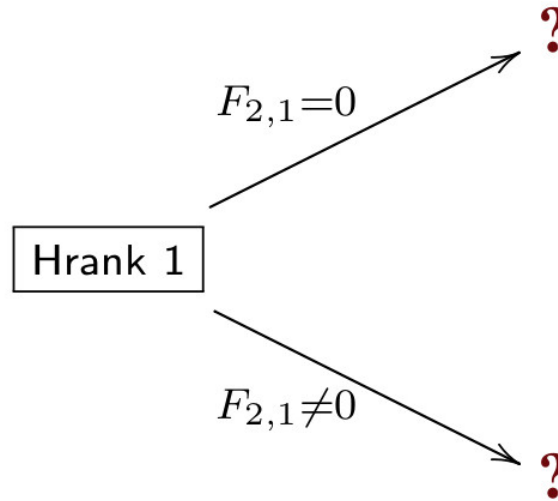
$$\stackrel{2,1}{=} -a_{1,1}^2 F_{2,1} + a_{1,1}^2 a_{2,2} G_{2,1},$$

shows that $G_{2,1}$ is a nonzero multiple of $F_{2,1}$. This means that $F_{2,1}$ is a *relative invariant*.

Consequently, if we abbreviate:

$$\boxed{\text{Hrank 1}} := \begin{array}{l} 0 \neq F_{xx} \\ 0 \equiv F_{xx} F_{yy} - F_{xy}^2 \end{array},$$

we must open two branches:



Proposition 1.5. *If a surface $S^2 \subset \mathbb{R}^3$ is affinely homogeneous and belongs to the branch $F_{2,1} = 0$, then $F = F(x)$ depends only on x , and the surface $S^2 = C^1 \times \mathbb{R}_y^1$ is a cylinder over a curve $C^1 := \{u = F(x)\}$ which is affinely homogeneous in \mathbb{R}^2 .*

Here and below, we will disregard such degenerate situations. That is, we will not attempt to expressly classify affinely homogeneous cylinders, because the task essentially boils down to lower dimension.

To prove this proposition, the key argument is to infinitesimalize and to exploit transitivity.

A general affine vector field writes:

$$\begin{aligned}
 L = & \left(T_1 + A_{1,1} x + A_{1,2} y + B_1 u \right) \frac{\partial}{\partial x} \\
 & + \left(T_2 + A_{2,1} x + A_{2,2} y + B_2 u \right) \frac{\partial}{\partial y} \\
 & + \left(T_0 + C_1 x + C_2 y + D u \right) \frac{\partial}{\partial u}.
 \end{aligned}$$

It is tangent to $\{u = F(x, y)\}$ if and only if:

$$\begin{aligned}
 0 & \equiv \text{eqL}(x, y) \\
 & =: L(-u + F(x, y)) \Big|_{u=F(x, y)},
 \end{aligned}$$

identically as power series in $\mathbb{R}\{x, y\}$. With increasing orders $\mu = 0, 1, 2, 3, \dots$, this eqL may be expanded:

$$\text{eqL} = \sum_{\mu=0}^{\infty} \sum_{i+j=\mu} \text{Coefficient}_{i,j} x^i y^j.$$

As for eqFG, denote:

$$[x^i y^j] \text{eqL} := \mathcal{C}_{i,j} = 0,$$

or shortly:

$$0 \stackrel{\mathbf{i},\mathbf{j}}{=} \mathcal{C}_{i,j}.$$

Such a vector field L is tangent to:

$$u = \frac{x^2}{2} + O_{x,y}(3),$$

if and only if:

$$0 \stackrel{\mathbf{0},\mathbf{0}}{=} -T_0,$$

$$0 \stackrel{\mathbf{1},\mathbf{0}}{=} -C_1 + T_1,$$

$$0 \stackrel{\mathbf{0},\mathbf{1}}{=} -C_2.$$

We then solve these 3 equations as:

$$T_0 := 0, \quad C_1 := T_1, \quad C_2 := 0.$$

In fact, the key constraint of transitivity:

$$\text{Span} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = T_{\text{origin}}S = \text{Span } L|_{\text{origin}} = \text{Span} \left(T_1 \frac{\partial}{\partial x} + T_2 \frac{\partial}{\partial y} \right),$$

forces to always keep T_1, T_2 absolutely free — never solved.

Next, such an L is tangent to:

$$u = \frac{x^2}{2} + F_{2,1} \frac{x^2 y}{2} + O_{x,y}(4),$$

if and only if moreover:

$$0 \stackrel{\mathbf{2,0}}{=} -\frac{1}{2} D + A_{1,1} + \frac{1}{2} F_{2,1} T_2,$$

$$0 \stackrel{\mathbf{1,1}}{=} A_{1,2} + F_{2,1} T_1,$$

$$0 \stackrel{\mathbf{0,2}}{=} 0.$$

We solve:

$$A_{1,2} := -F_{2,1} T_1,$$

$$D := F_{2,1} T_2 + 2 A_{1,1}.$$

Proof of Proposition 1.5. Since $F_{2,1} = 0$ is assumed, we have by letting order 4 monomials appear:

$$u = \frac{x^2}{2} + \mathbf{0} + F_{4,0} \frac{x^4}{24} + F_{3,1} \frac{x^3 y}{6} + F_{2,2} \frac{x^2 y^2}{4} + F_{1,3} \frac{x y^3}{6} + F_{0,4} \frac{y^4}{24} + O_{x,y}(5),$$

and:

$$A_{1,2} = 0,$$

$$D = 0 + 2 A_{1,1}.$$

Then at order 3, eqL gives:

$$\begin{aligned}
 0 &\stackrel{\mathbf{3,0}}{=} \frac{1}{6} F_{4,0} T_1 + \frac{1}{2} B_1 + \frac{1}{6} F_{3,1} T_2, \\
 0 &\stackrel{\mathbf{2,1}}{=} \frac{1}{2} F_{3,1} T_1 + \frac{1}{2} F_{2,2} T_2, \\
 0 &\stackrel{\mathbf{1,2}}{=} \frac{1}{2} F_{2,2} T_1 + \frac{1}{2} F_{1,3} T_2, \\
 0 &\stackrel{\mathbf{0,3}}{=} \frac{1}{6} F_{1,3} T_1 + \frac{1}{6} F_{0,4} T_2.
 \end{aligned}$$

Since there can be no linear relation between the transitivity parameters $\{T_1, T_2\}$, we necessarily have:

$$0 = F_{0,4} = F_{1,3} = F_{2,2} = F_{3,1}.$$

Thus:

$$\begin{aligned}
 u &= \frac{x^2}{2} + \mathbf{0} + F_{4,0} \frac{x^4}{24} \\
 &+ F_{5,0} \frac{x^5}{120} + F_{4,1} \frac{x^4 y}{24} + F_{3,2} \frac{x^3 y^2}{12} + F_{2,3} \frac{x^2 y^3}{12} + F_{1,4} \frac{x y^4}{24} + F_{0,5} \frac{y^5}{120} \\
 &+ O_{x,y}(6).
 \end{aligned}$$

Again, eqL at order 4 gives:

$$\begin{aligned}
 0 &\stackrel{\mathbf{4,0}}{=} \frac{1}{12} F_{4,0} A_{1,1} + \frac{1}{24} F_{5,0} T_1 + \frac{1}{24} F_{4,1} T_2, \\
 0 &\stackrel{\mathbf{3,1}}{=} \frac{1}{6} F_{4,1} T_1 + \frac{1}{6} F_{3,2} T_2, \\
 0 &\stackrel{\mathbf{2,2}}{=} \frac{1}{4} F_{3,2} T_1 + \frac{1}{4} F_{2,3} T_2, \\
 0 &\stackrel{\mathbf{1,3}}{=} \frac{1}{6} F_{2,3} T_1 + \frac{1}{6} F_{1,4} T_2, \\
 0 &\stackrel{\mathbf{0,4}}{=} \frac{1}{24} F_{1,4} T_1 + \frac{1}{24} F_{0,5} T_2.
 \end{aligned}$$

By freeness of $\{T_1, T_2\}$, it is necessary that:

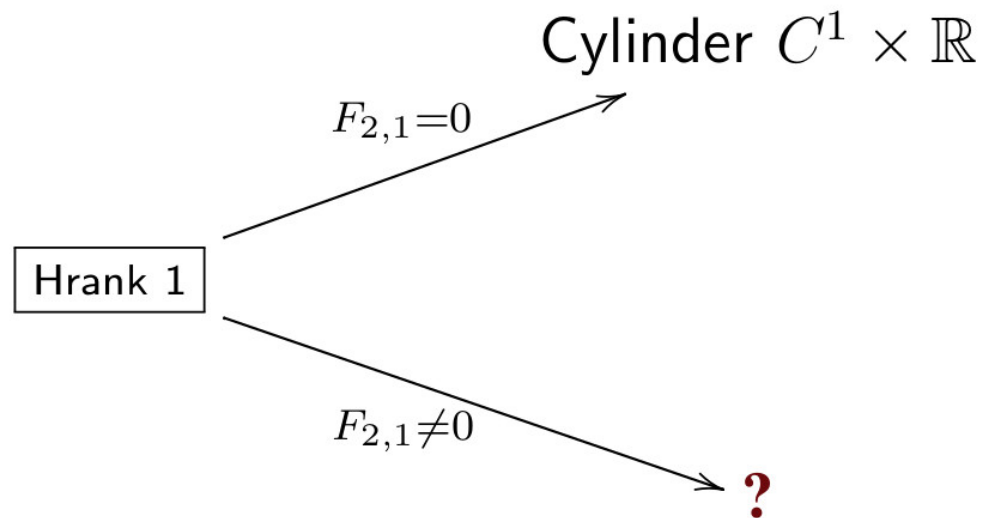
$$0 = F_{0,5} = F_{1,4} = F_{2,3} = F_{3,2} = F_{4,1}.$$

An elementary induction on the order $\mu \geq 6$ shows that in the expansion:

$$u = \frac{x^2}{2} + \mathbf{0} + F_{4,0} \frac{x^4}{24} + F_{5,0} \frac{x^5}{120} + \sum_{\mu=6}^{\infty} \sum_{i+j=\mu} F_{i,j} \frac{x^i}{i!} \frac{y^j}{j!},$$

all $F_{i,j}$ with $j \geq 1$ must be zero, so that $F = F(x)$ is in conclusion independent of y .

Lastly, it can be verified that affine homogeneity in \mathbb{R}^3 of the cylindrical surface $\{(x, y, u) : u = F(x)\}$ is equivalent to affine homogeneity in \mathbb{R}^2 of the curve $\{(x, u) : u = F(x)\}$. \square



The branch $F_{2,1} = 0$ being thus settled, assume $F_{2,1} \neq 0$. Since $F_{2,1} \propto G_{2,1}$, this is a *coordinate-independent assumption*. Indeed, recall:

$$\stackrel{\mathbf{2,1}}{=} -a_{1,1}^2 F_{2,1} + a_{1,1}^2 a_{2,2} G_{2,1},$$

with $a_{1,1} \neq 0 \neq a_{2,2}$ thanks to:

$$0 \neq \begin{vmatrix} a_{1,1} & 0 & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ 0 & 0 & a_{1,1}^2 \end{vmatrix} = a_{1,1} a_{2,2} a_{1,1}^2.$$

In this equation ^{2,1}, it is clear that one can normalize $G_{2,1} := 1$ by choosing $a_{2,2} := F_{2,1}$.

In accordance with Principle 1.4, restart, rename $G := F$ with $F_{2,1} = 1$, and normalize similarly $G_{2,1} := 1$. Thus:

$$\begin{array}{l} u = \frac{x^2}{2} + \frac{x^2 y}{2} + O_{x,y}(4) \\ \xrightarrow{\text{Equivalence}} v = \frac{r^2}{2} + \frac{r^2 s}{2} + O_{r,s}(4). \end{array}$$

Lemma 1.6. *Stabilization of these order ≤ 3 normalizations holds if and only if:*

$$\begin{bmatrix} a_{1,1} & \mathbf{0} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{2}} \rightsquigarrow \begin{bmatrix} a_{1,1} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & \mathbf{1} & b_2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{3}}.$$

Proof. Examine eqFG at order 3:

$$0 \stackrel{\mathbf{3,0}}{=} \frac{1}{2} a_{1,1} b_1 + \frac{1}{2} a_{1,1}^2 a_{2,1},$$

$$0 \stackrel{\mathbf{2,1}}{=} -\frac{1}{2} a_{1,1}^2 + \frac{1}{2} a_{1,1}^2 a_{2,2},$$

$$0 \stackrel{\mathbf{1,2}}{=} 0,$$

$$0 \stackrel{\mathbf{0,3}}{=} 0. \quad \square$$

Next, pass to order 4:

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + F_{4,0} \frac{x^4}{24} + F_{3,1} \frac{x^3 y}{6} + \underline{\frac{x^2 y^2}{2}} + 0 + 0 + O_{x,y}(5),$$

$$v = \frac{r^2}{2} + \frac{r^2 s}{2} + G_{4,0} \frac{r^4}{24} + G_{3,1} \frac{r^3 s}{6} + \underline{\frac{r^2 s^2}{2}} + 0 + 0 + O_{r,s}(5).$$

Here, the values of the underlined monomials are obtained from the (affinely invariant) hypothesis of constant Hessian rank 1.

Indeed, from:

$$F_{yy} \equiv \frac{F_{xy}^2}{F_{xx}} \iff G_{ss} \equiv \frac{G_{rs}^2}{G_{rr}},$$

by successive differentiations and replacement, taking values at the origin, one convinces oneself (*see* also arxiv.org/abs/2010.02873/), that all

$F_{i,j}$ with $j \geq 2$ express in terms of the $F_{j',0}$ with $j' \leq i + j$ and of the $F_{j',1}$ with $j' + 1 \leq i + j$. Here, one obtains $F_{2,2} = 2$, $F_{1,3} = 0$, $F_{0,4} = 0$, and the same for G .

Lemma 1.7. *One can normalize $G_{4,0} := 0$.*

Proof. Indeed, eqFG gives, with the free parameter b_2 :

$$\begin{aligned}
 0 &\stackrel{\mathbf{4,0}}{=} -\frac{1}{24} a_{1,1}^2 F_{4,0} + \frac{1}{24} a_{1,1}^4 G_{4,0} + \frac{1}{6} a_{1,1}^3 a_{2,1} G_{3,1} + \frac{1}{8} a_{1,1}^2 a_{2,1}^2 + \frac{1}{4} a_{1,1}^2 \boxed{b_2}, \\
 0 &\stackrel{\mathbf{3,1}}{=} -\frac{1}{6} a_{1,1}^2 F_{3,1} + \frac{1}{6} a_{1,1}^3 G_{3,1}. \quad \square
 \end{aligned}$$

Visibly, $G_{3,1} \propto F_{3,1}$ is a relative invariant, and we have:

$$\begin{aligned}
 u &= \frac{x^2}{2} + \frac{x^2 y}{2} + F_{3,1} \frac{x^3 y}{6} + \frac{x^2 y^2}{2} + O_{x,y}(5) \\
 \xrightarrow{\text{Equivalence}} v &= \frac{r^2}{2} + \frac{r^2 s}{2} + G_{3,1} \frac{r^3 s}{6} + \frac{r^2 s^2}{4} + O_{r,s}(4).
 \end{aligned}$$

Lemma 1.8. *Stabilization of this order ≤ 4 normalization holds if and only if:*

$$\begin{bmatrix} a_{1,1} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & \mathbf{1} & b_2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{3}} \rightsquigarrow \begin{bmatrix} a_{1,1} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & \mathbf{1} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{1,1}a_{2,1}G_{3,1} \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{4}}.$$

Proof. After setting $G_{4,0} := 0 =: F_{4,0}$, solve b_2 in equation $\stackrel{\mathbf{4,0}}{=}$ above. \square

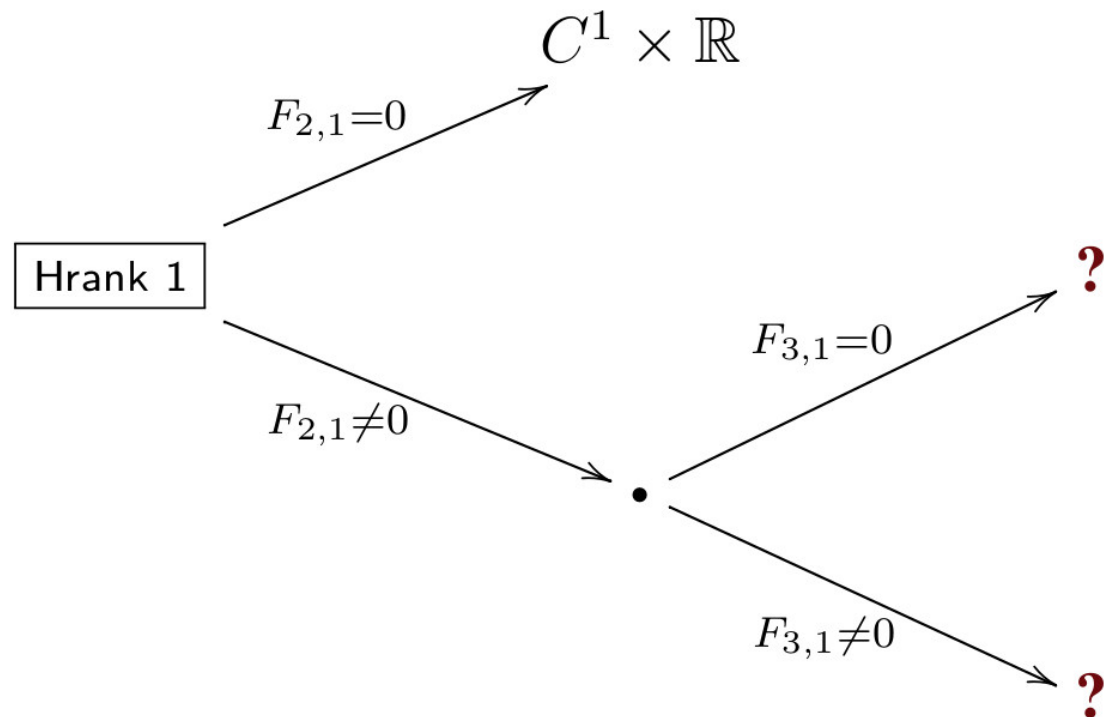
Coming back to order 3, in the infinitesimal counterpart eqL:

$$\begin{aligned} 0 &\stackrel{\mathbf{3,0}}{=} \frac{1}{6} F_{3,1} T_2 + \frac{1}{2} A_{2,1} + \frac{1}{2} B_1, \\ 0 &\stackrel{\mathbf{2,1}}{=} \frac{1}{2} F_{3,1} T_1 + \frac{1}{2} T_2 + \frac{1}{2} A_{2,2}, \end{aligned}$$

we normalize:

$$\begin{aligned} B_1 &:= -\frac{1}{3} F_{3,1} T_2 - A_{2,1}, \\ A_{2,2} &= -F_{3,1} T_1 - T_2. \end{aligned}$$

Beyond, because $F_{3,1}$ is a relative invariant, we must open two branches:



We study first the branch $F_{3,1} = 0$, and we let terms of order 5 appear:

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^2 y^2}{2} + F_{5,0} \frac{x^5}{120} + F_{4,1} \frac{x^4 y}{24} + \frac{x^2 y^3}{2} + O_{x,y}(6).$$

Lemma 1.9. *In the branch $F_{3,1} = 0$, affine homogeneity forces $F_{4,1} = 0$, necessarily.*

Proof. Indeed, eqL gives:

$$0 \stackrel{\mathbf{4,0}}{=} \frac{1}{24} F_{5,0} T_1 + \frac{1}{24} F_{4,1} T_2 + \frac{1}{4} B_2,$$

$$0 \stackrel{\mathbf{3,1}}{=} \frac{1}{6} F_{4,1} T_1. \quad \square$$

Next, solve:

$$B_2 := -\frac{1}{6} F_{5,0} T_1 - 0.$$

Thus with $F_{4,1} := 0 =: G_{4,1}$:

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^2 y^2}{2} + F_{5,0} \frac{x^5}{120} + 0 + \frac{x^2 y^3}{2} + O_{x,y}(6),$$

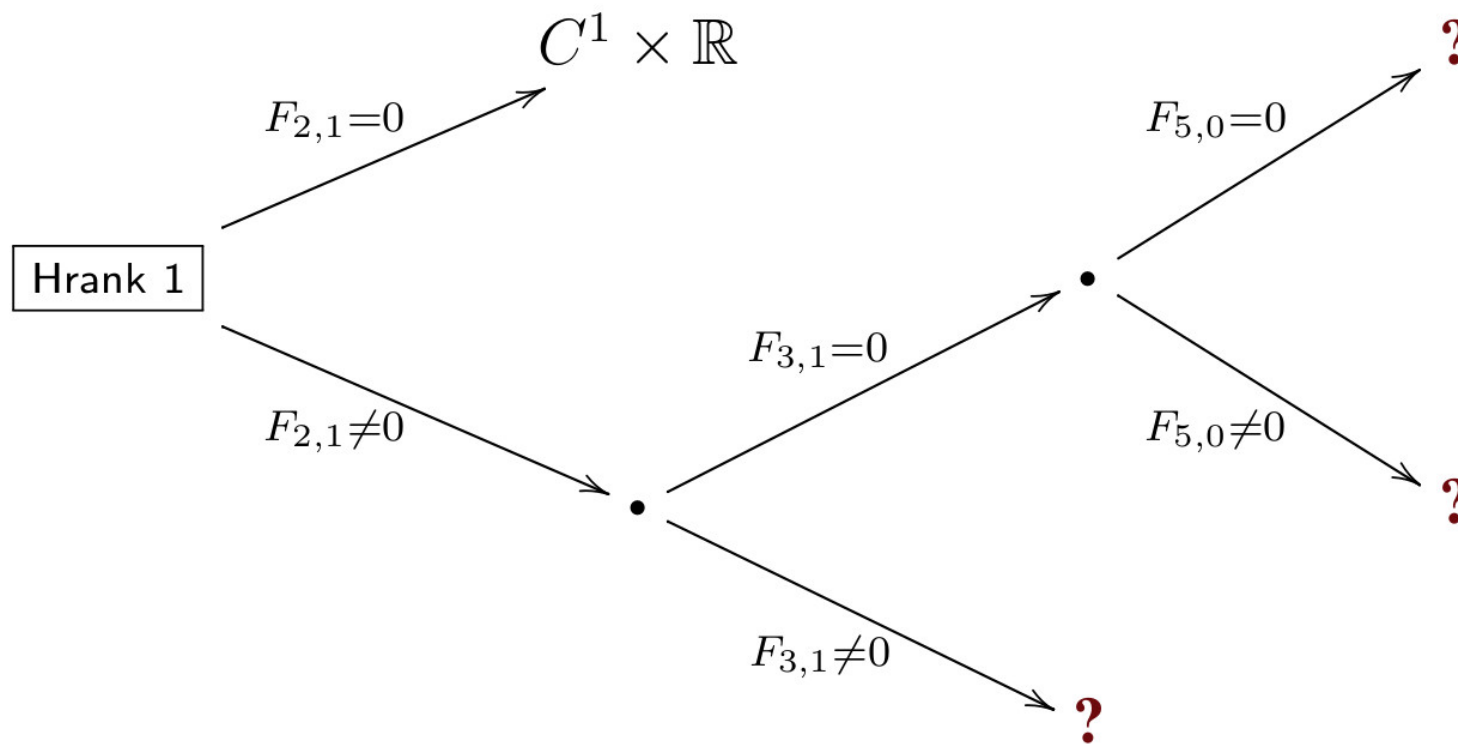
$\xrightarrow{\text{Equivalence}}$

$$v = \frac{r^2}{2} + \frac{r^2 s}{2} + \frac{r^2 s^2}{2} + G_{5,0} \frac{r^5}{120} + 0 + \frac{r^2 s^3}{2} + O_{r,s}(6),$$

With this, eqFG at order $\mu = 5$ contains only one nonzero equation:

$$0 \stackrel{\mathbf{5,0}}{=} -\frac{1}{120} F_{5,0} a_{1,1}^2 + \frac{1}{120} G_{5,0} a_{1,1}^5.$$

Therefore, $F_{5,0} \propto G_{5,0}$ is a relative invariant: it creates a new branching:



Study first the subbranch $F_{5,0} = 0$:

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^2 y^2}{2} + 0 + 0 + \frac{x^2 y^3}{2} + O_{x,y}(6),$$

with $v = G$ similarly given. At order 4, the isotropy group from Lemma 1.8:

$$\begin{bmatrix} a_{1,1} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & \mathbf{1} & -\frac{1}{2}a_{2,1}^2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^4,$$

is still 2-dimensional, with parameters $a_{1,1}, a_{2,1}$.

Proposition 1.10. *In the branch $F_{2,1} \neq 0, F_{3,1} = 0, F_{5,0} = 0$, if the surface $S^2 \subset \mathbb{C}^3$ is affinely homogeneous, then all $F_{j,k} = 0$ except $F_{2,k} = k!$ for every $k = 1, 2, 3, 4, 5, \dots$*

Proof. Examine eqL at order 5:

$$0 \stackrel{\mathbf{5,0}}{=} \frac{1}{120} F_{6,0} T_1 + \frac{1}{120} F_{5,1} T_2,$$

$$0 \stackrel{\mathbf{4,1}}{=} \frac{1}{24} F_{5,1} T_1 + \frac{1}{24} F_{4,2} T_2,$$

$$0 \stackrel{\mathbf{3,2}}{=} \frac{1}{12} F_{4,2} T_1 + \frac{1}{12} F_{3,3} T_2,$$

$$0 \stackrel{\mathbf{2,3}}{=} \frac{1}{12} F_{3,3} T_1 + \left(-2 + \frac{1}{12} F_{2,4} \right) T_2,$$

$$0 \stackrel{\mathbf{1,4}}{=} \left(\frac{1}{24} F_{2,4} - 1 \right) T_1 + \frac{1}{24} F_{1,5} T_2,$$

$$0 \stackrel{\mathbf{0,5}}{=} \frac{1}{120} F_{1,5} T_1 + \frac{1}{120} F_{0,6} T_2,$$

to get:

$$F_{2,4} = 4!,$$

$$0 = F_{0,6} = F_{1,5} = F_{3,3} = F_{4,2} = F_{5,1} = F_{6,0}.$$

Next, eqL at order 6:

$$\begin{aligned}
0 &\stackrel{\mathbf{6,0}}{=} \frac{1}{720} F_{7,0} T_1 + \frac{1}{720} F_{6,1} T_2, \\
0 &\stackrel{\mathbf{5,1}}{=} \frac{1}{120} F_{6,1} T_1 + \frac{1}{120} F_{5,2} T_2, \\
0 &\stackrel{\mathbf{4,2}}{=} \frac{1}{48} F_{5,2} T_1 + \frac{1}{48} F_{4,3} T_2, \\
0 &\stackrel{\mathbf{3,3}}{=} \frac{1}{36} F_{4,3} T_1 + \frac{1}{36} F_{3,4} T_2, \\
0 &\stackrel{\mathbf{2,4}}{=} \frac{1}{48} F_{3,4} T_1 + \left(-\frac{5}{2} + \frac{1}{48} F_{2,5} \right) T_2, \\
0 &\stackrel{\mathbf{1,5}}{=} \left(\frac{1}{120} F_{2,5} - 1 \right) T_1 + \frac{1}{120} F_{1,6} T_1, \\
0 &\stackrel{\mathbf{0,6}}{=} \frac{1}{720} F_{1,6} T_1 + \frac{1}{720} F_{0,7} T_2.
\end{aligned}$$

solve similarly:

$$F_{2,5} = 5!,$$

$$0 = F_{0,7} = F_{1,6} = F_{3,4} = F_{4,3} = F_{5,2} = F_{6,1} = F_{7,0}.$$

An induction on the order $\mu = i + j$ is elementary. □

Since $\sum_k y^k = \frac{1}{1-y}$, we obtain

Theorem 1.11. *In the branch $F_{2,1} \neq 0$, $F_{3,1} = 0$, $F_{5,0} = 0$, there is a single affinely homogeneous surface $S^2 \subset \mathbb{R}^3$:*

$$u = \frac{1}{2} \frac{x^2}{1-y},$$

which has 4-dimensional transitive affine Lie symmetry algebra generated by:

$$e_1 := (1-y) \partial_x + x \partial_u,$$

$$e_2 := (1-y) \partial_y + u \partial_u,$$

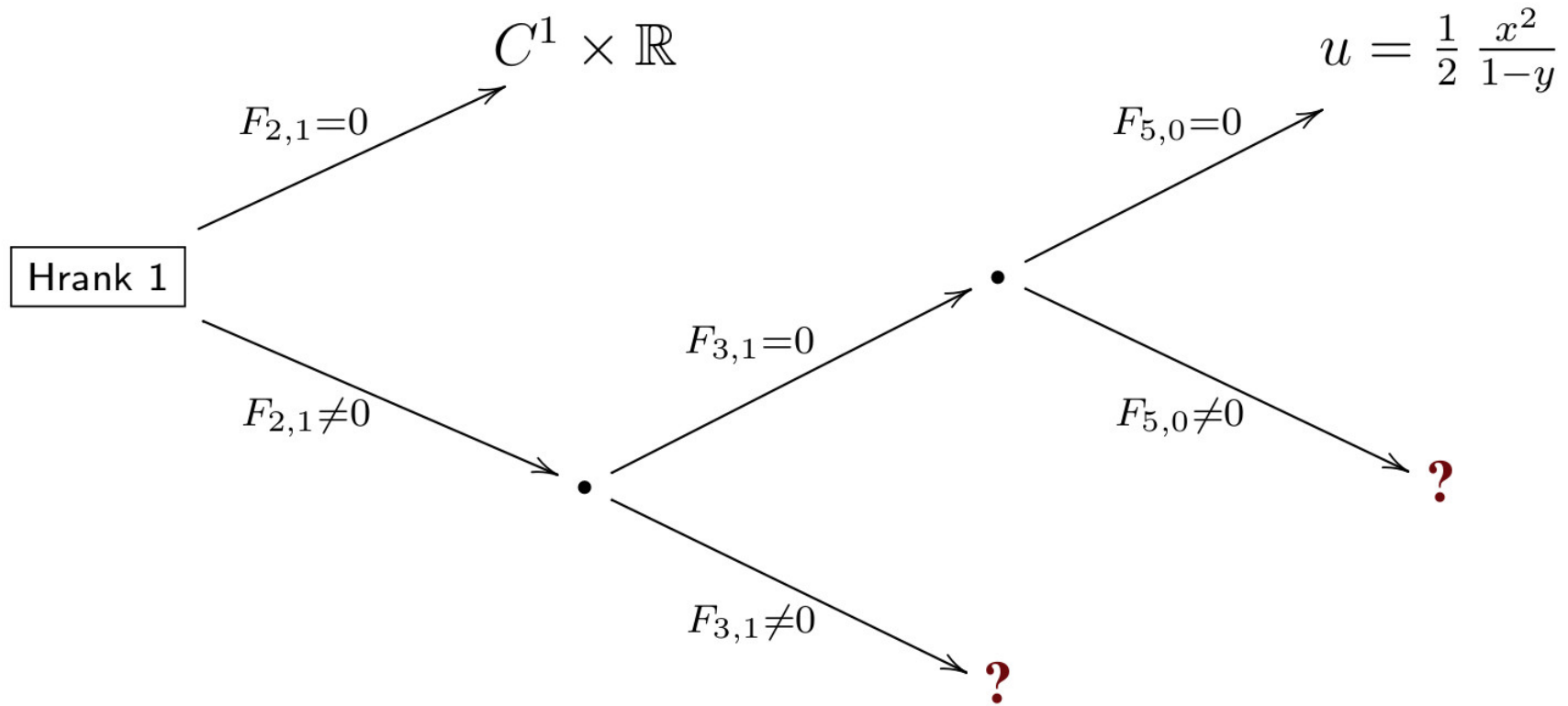
$$e_3 := x \partial_x + 2u \partial_u,$$

$$e_4 := -u \partial_x + x \partial_y,$$

sharing the Lie brackets:

$$[e_1, e_2] = e_1, [e_1, e_3] = e_1, [e_1, e_4] = e_2, [e_2, e_4] = e_4, [e_3, e_4] = e_4.$$

□



Next, let us study the subbranch $F_{5,0} \neq 0$. From:

$$0 \stackrel{5,0}{=} -\frac{1}{120} F_{5,0} a_{1,1}^2 + \frac{1}{120} G_{5,0} a_{1,1}^5,$$

taking $a_{1,1} := \sqrt[3]{F_{5,0}}$, we normalize $G_{5,0} := 1 =: F_{5,0}$. To stabilize:

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^2 y^2}{2} + \frac{x^5}{120} + \frac{x^2 y^3}{2} + O_{x,y}(6)$$

Equivalence
 $\xrightarrow{\hspace{1.5cm}}$

$$v = \frac{r^2}{2} + \frac{r^2 s}{2} + \frac{r^2 s^2}{2} + \frac{r^5}{120} + \frac{r^2 s^3}{2} + O_{r,s}(6),$$

we need to satisfy:

$$0 \stackrel{5,0}{=} -\frac{1}{120} a_{1,1}^2 + \frac{1}{120} a_{1,1}^5,$$

and we set $a_{1,1} := 1$.

At the infinitesimal level, eqL for order 5 gives:

$$0 \stackrel{5,0}{=} \frac{1}{120} F_{6,0} T_1 + \left(-\frac{1}{120} + \frac{1}{120} F_{5,1} \right) T_2 + \frac{1}{40} A_{1,1},$$

$$0 \stackrel{4,1}{=} \left(\frac{1}{24} F_{5,1} - \frac{1}{6} \right) T_1 + \frac{1}{24} F_{4,2} T_2,$$

$$0 \stackrel{3,2}{=} \frac{1}{12} F_{4,2} T_1 + \frac{1}{12} F_{3,3} T_2,$$

$$0 \stackrel{2,3}{=} \frac{1}{12} F_{3,3} T_1 + \left(-2 + \frac{1}{12} F_{2,4} \right) T_2,$$

$$0 \stackrel{1,4}{=} \left(\frac{1}{24} F_{2,4} - 1 \right) T_1 + \frac{1}{24} F_{1,5} T_2,$$

$$0 \stackrel{0,5}{=} \frac{1}{120} F_{1,5} T_1 + \frac{1}{120} F_{0,6} T_2,$$

whence:

$$F_{5,1} = 4, \quad F_{2,4} = 4!,$$

$$0 = F_{0,6} = F_{1,5} = F_{3,3} = F_{4,2},$$

and lastly:

$$A_{1,1} := -\frac{1}{3} F_{6,0} T_1 - T_2.$$

Next, at order 6, putting similarly as always:

$$G_{5,1} = 4, \quad G_{2,4} = 4!, \\ 0 = G_{0,6} = G_{1,5} = G_{3,3} = G_{4,2},$$

only one nontrivial equation exists:

$$0 \stackrel{\mathbf{6,0}}{=} -\frac{1}{720} F_{6,0} + \frac{1}{720} G_{6,0} + \frac{1}{240} \boxed{a_{2,1}}.$$

Using $a_{2,1}$, we normalize:

$$G_{6,0} := 0 =: F_{6,0},$$

and then we stabilize:

$$a_{2,1} := 0.$$

Since the isotropy matrix is now reduced to the identity:

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix},$$

eqFG is terminated, and only eqL must be examined further.

At order 6, eqL gives:

$$0 \stackrel{\mathbf{6,0}}{=} \frac{1}{720} F_{7,0} T_1 + \frac{1}{720} F_{6,1} T_2 + \frac{1}{240} A_{2,1},$$

$$0 \stackrel{\mathbf{5,1}}{=} \frac{1}{120} F_{6,1} T_1 + \left(-\frac{1}{6} + \frac{1}{120} F_{5,2} \right) T_2,$$

$$0 \stackrel{\mathbf{4,2}}{=} \left(\frac{1}{48} F_{5,2} - \frac{5}{12} \right) T_1 + \frac{1}{48} F_{4,3} T_2,$$

$$0 \stackrel{\mathbf{3,3}}{=} \frac{1}{36} F_{4,3} T_1 + \frac{1}{36} F_{3,4} T_2,$$

$$0 \stackrel{\mathbf{2,4}}{=} \frac{1}{48} F_{3,4} T_1 + \left(-\frac{5}{2} + \frac{1}{48} F_{2,5} \right) T_2,$$

$$0 \stackrel{\mathbf{1,5}}{=} \left(\frac{1}{120} F_{2,5} - 1 \right) T_1 + \frac{1}{120} F_{1,6} T_2,$$

$$0 \stackrel{\mathbf{0,6}}{=} \frac{1}{720} F_{1,6} T_1 + \frac{1}{720} F_{0,7} T_2,$$

whence:

$$\begin{aligned} F_{0,7} &:= 0, & F_{1,6} &:= 0, & F_{2,5} &:= 120, \\ F_{3,4} &:= 0, & F_{4,3} &:= 0, & F_{5,2} &:= 20, & F_{6,1} &:= 0, \end{aligned}$$

and lastly:

$$A_{2,1} := -\frac{1}{3} F_{7,0} T_1.$$

At order 7, eqL gives:

$$0 \stackrel{7,0}{=} \left(\frac{1}{5040} F_{8,0} - \frac{1}{288} \right) T_1 + \left(-\frac{1}{840} F_{7,0} + \frac{1}{5040} F_{7,1} \right) T_2,$$

$$0 \stackrel{6,1}{=} \left(\frac{1}{720} F_{7,1} - \frac{1}{120} F_{7,0} \right) T_1 + \frac{1}{720} F_{6,2} T_2,$$

$$0 \stackrel{5,2}{=} \frac{1}{240} F_{6,2} T_1 + \left(-\frac{1}{2} + \frac{1}{240} F_{5,3} \right) T_2,$$

$$0 \stackrel{4,3}{=} \left(\frac{1}{144} F_{5,3} - \frac{5}{6} \right) T_1 + \frac{1}{144} F_{4,4} T_2,$$

$$0 \stackrel{3,4}{=} \frac{1}{144} F_{4,4} T_1 + \frac{1}{144} F_{3,5} T_2,$$

$$0 \stackrel{2,5}{=} \frac{1}{240} F_{3,5} T_1 + \left(-3 + \frac{1}{240} F_{2,6} \right) T_2,$$

$$0 \stackrel{1,6}{=} \left(\frac{1}{720} F_{2,6} - 1 \right) T_1 + \frac{1}{720} F_{1,7} T_2,$$

$$0 \stackrel{0,7}{=} \frac{1}{5040} F_{1,7} T_1 + \frac{1}{5040} F_{0,8} T_2,$$

which is solved as:

$$F_{0,8} := 0, \quad F_{1,7} := 0, \quad F_{2,6} := 720, \quad F_{3,5} := 0, \quad F_{4,4} := 0,$$

$$F_{5,3} := 120, \quad F_{6,2} := 0, \quad F_{7,1} := 6 F_{7,0}, \quad F_{8,0} := \frac{35}{2},$$

with an invariant:

$$F_{7,0} =: \theta \in \mathbb{R},$$

which may take any real value.

At order 8, the resolution of the (unwritten) equations of eqL is:

$$F_{0,9} := 0, \quad F_{1,8} := 0, \quad F_{2,7} := 5040, \quad F_{3,6} := 0, \quad F_{4,5} := 0, \\ F_{5,4} := 840, \quad F_{6,3} := 0, \quad F_{7,2} := 42\theta, \quad F_{8,1} := \frac{245}{2}, \quad F_{9,0} := 4\theta^2.$$

One therefore finds a 1-parameter family of affinely inequivalent homogeneous models $(S_\theta^2)_{\theta \in \mathbb{R}}$.

Proposition 1.12. *In the branch $F_{2,1} \neq 0$, $F_{3,1} = 0$, $F_{5,0} \neq 0$, there is a 1-parameter family of inequivalent affinely homogeneous surfaces $S_\theta^2 \subset \mathbb{R}^3$:*

$$u = \frac{1}{2}x^2 + \frac{1}{2}x^2y + \frac{1}{2}x^2y^2 \\ + \frac{1}{120}x^5 + \frac{1}{2}x^2y^3 \\ + \frac{1}{30}x^5y + \frac{1}{2}x^2y^4 \\ + \frac{1}{5040}\theta x^7 + \frac{1}{12}x^5y^2 + \frac{1}{2}x^2y^5 \\ + \frac{1}{2304}x^8 + \frac{1}{840}\theta x^7y + \frac{1}{6}x^5y^3 + \frac{1}{2}x^2y^6 \\ + \frac{1}{90720}\theta^2 x^9 + \frac{7}{2304}x^8y + \frac{1}{240}\theta x^7y^2 + \frac{7}{24}x^5y^4 + \frac{1}{2}x^2y^7 + O_{x,y}(10).$$

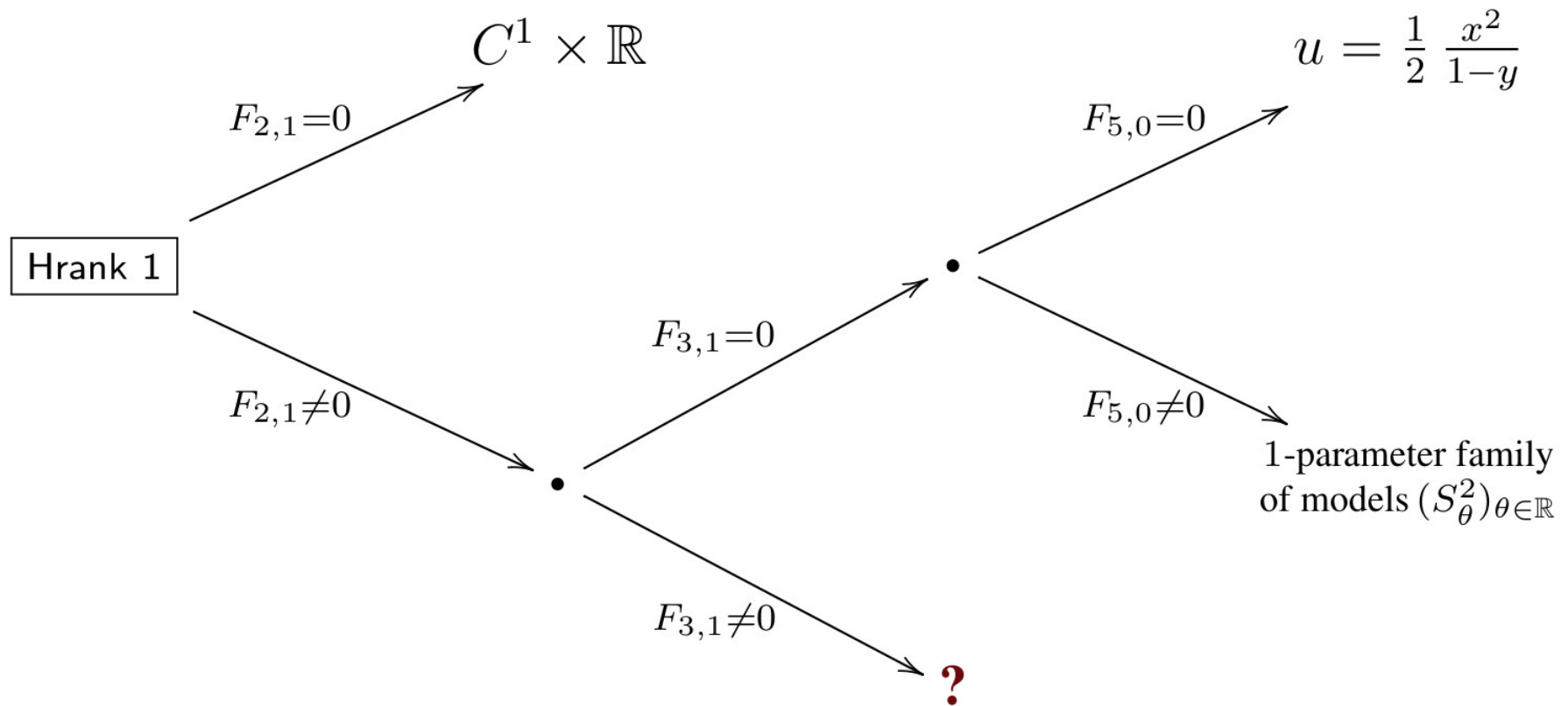
with 2-dimensional (simply transitive) commutative affine Lie symmetry algebra:

$$e_1 := \left(1 - y + \frac{1}{3} \theta u\right) \partial_x + \left(-\frac{1}{3} \theta x - \frac{1}{6} u\right) \partial_y + x \partial_u,$$

$$e_2 := -x \partial_x + (1 - y) \partial_y - u \partial_u,$$

$$[e_1, e_2] = 0.$$

□



It remains to explore the subbranch $F_{3,1} \neq 0$, within the branch $F_{2,1} = 1$. From the proof of Lemma 1.7:

$$0 \stackrel{\mathbf{3,1}}{=} -\frac{1}{6} a_{1,1}^2 F_{3,1} + \frac{1}{6} a_{1,1}^3 G_{3,1},$$

it is clear that we can normalize $G_{3,1} := 1 =: F_{3,1}$. Thus:

$$\begin{aligned} u &= \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^3 y}{6} + \frac{x^2 y^2}{2} + O_{x,y}(5) \\ \xrightarrow{\text{Equivalence}} v &= \frac{r^2}{2} + \frac{r^2 s}{2} + \frac{r^3 s}{6} + \frac{r^2 s^2}{2} + O_{r,s}(5). \end{aligned}$$

Since $\stackrel{\mathbf{3,1}}{=}$ becomes $0 = -\frac{1}{6} a_{1,1}^2 + \frac{1}{6} a_{1,1}^3$, we set $a_{1,1} := 1$, hence the stability group at orders ≤ 4 is 1-dimensional:

$$\left[\begin{array}{ccc} \mathbf{1} & \mathbf{0} & -a_{2,1} \\ a_{2,1} & \mathbf{1} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{2,1} \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{array} \right]^{\mathbf{4}}.$$

Next, eqL at order 4 gives:

$$\begin{aligned}
0 &\stackrel{4,0}{=} \frac{1}{24} F_{5,0} T_1 + \frac{1}{24} F_{4,1} T_2 + \frac{1}{6} A_{2,1} + \frac{1}{4} B_2, \\
0 &\stackrel{3,1}{=} \left(\frac{1}{6} F_{4,1} - \frac{1}{6}\right) T_1 + \left(-\frac{2}{3} + \frac{1}{6} F_{3,2}\right) T_2, \\
0 &\stackrel{2,2}{=} \left(\frac{1}{4} F_{3,2} - \frac{3}{2}\right) T_1 + \left(-\frac{3}{2} + \frac{1}{4} F_{2,3}\right) T_2, \\
0 &\stackrel{1,3}{=} \left(\frac{1}{6} F_{2,3} - 1\right) T_1 + \frac{1}{6} F_{1,4} T_2, \\
0 &\stackrel{0,4}{=} \frac{1}{24} F_{1,4} T_1 + \frac{1}{24} F_{0,5} T_2,
\end{aligned}$$

whence:

$$0 = F_{0,5} = F_{1,4}, \quad F_{2,3} = 6, \quad F_{3,2} = 6,$$

and:

$$\begin{aligned}
A_{1,1} &:= \left(-F_{4,1} + 1\right) T_1 - 2 T_2, \\
B_2 &:= -\frac{1}{6} F_{5,0} T_1 - \frac{1}{6} F_{4,1} T_2 - \frac{2}{3} A_{2,1}.
\end{aligned}$$

Putting in eqFG at order 5:

$$0 = G_{0,5} = G_{1,4}, \quad G_{2,3} = 6, \quad G_{3,2} = 6,$$

we get:

$$0 \stackrel{\mathbf{5,0}}{=} -\frac{1}{120} F_{5,0} + \frac{1}{120} G_{5,0} + \frac{1}{24} G_{4,1} a_{2,1} - \frac{1}{18} a_{2,1} + \frac{1}{24} a_{2,1}^2,$$

$$0 \stackrel{\mathbf{4,1}}{=} -\frac{1}{24} F_{4,1} + \frac{1}{24} G_{4,1} + \frac{1}{12} \boxed{a_{2,1}}.$$

Using the last remaining group parameter $a_{2,1}$, we normalize $G_{4,1} := 0 =: F_{4,1}$, whence $a_{2,1} := 0$ so that the group reduction descends to identity:

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & -a_{2,1} \\ a_{2,1} & \mathbf{1} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{2,1} \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{4}} \rightsquigarrow \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathbf{5}},$$

to stabilize the normal forms at orders ≤ 5 :

$$u = \frac{x^2}{2} + \frac{x^2y}{2} + \frac{x^3y}{6} + \frac{x^2y^2}{2} + \frac{1}{120} F_{5,0} x^5 + \frac{1}{2} x^3y^2 + \frac{1}{2} x^2y^3 + O_{x,y}(6)$$

$$\xrightarrow{\text{Equivalence}} v = \frac{r^2}{2} + \frac{r^2s}{2} + \frac{r^3s}{6} + \frac{r^2s^2}{2} + \frac{1}{120} G_{5,0} r^5 + \frac{1}{2} r^3s^2 + \frac{1}{2} s^2r^3 + O_{r,s}(6).$$

Thus, eqFG is terminated, and only simply transitive homogeneous models can be found in this branch. The only remaining equation:

$$0 \stackrel{\mathbf{5,0}}{=} -\frac{1}{120} F_{5,0} + \frac{1}{120} G_{5,0},$$

seems to say that $F_{5,0}$ is an absolute invariant which may take any value $\eta \in \mathbb{R}$, but we will see at higher orders that only 1 specific numeric value is possible for $F_{5,0}$.

Next, go to eqL at order 5:

$$0 \stackrel{\mathbf{5,0}}{=} \left(\frac{1}{90} F_{5,0} + \frac{1}{120} F_{6,0} \right) T_1 + \left(-\frac{7}{120} F_{5,0} + \frac{1}{120} F_{5,1} \right) T_2 - \frac{1}{18} A_{2,1},$$

$$0 \stackrel{\mathbf{4,1}}{=} \left(\frac{1}{24} F_{5,1} - \frac{1}{6} F_{5,0} \right) T_1 + \left(-\frac{5}{36} + \frac{1}{24} F_{4,2} \right) T_2 + \frac{1}{12} A_{2,1},$$

$$0 \stackrel{\mathbf{3,2}}{=} \left(-\frac{1}{2} + \frac{1}{12} F_{4,2} \right) T_1 + \left(-3 + \frac{1}{12} F_{3,3} \right) T_2,$$

$$0 \stackrel{\mathbf{2,3}}{=} \left(-3 + \frac{1}{12} F_{3,3} \right) T_1 + \left(-2 + \frac{1}{12} F_{2,4} \right) T_2,$$

$$0 \stackrel{\mathbf{1,4}}{=} \left(\frac{1}{24} F_{2,4} - 1 \right) T_1 + \frac{1}{24} F_{1,5} T_2,$$

$$0 \stackrel{\mathbf{0,5}}{=} \frac{1}{120} F_{1,5} T_1 + \frac{1}{120} F_{0,6} T_2.$$

Firstly, solve the last four equations:

$$F_{0,6} := 0, \quad F_{1,5} := 0, \quad F_{2,4} := 24, \quad F_{3,3} := 36, \quad F_{4,2} := 6,$$

and secondly, solve $\stackrel{\mathbf{4,1}}{=}$:

$$A_{2,1} := \left(-\frac{1}{2} F_{5,1} + 2 F_{5,0} \right) T_1 - \frac{4}{3} T_2.$$

There remains one equation:

$$0 \stackrel{5,0}{=} \left(-\frac{1}{10} F_{5,0} + \frac{1}{120} F_{6,0} + \frac{1}{36} F_{5,1} \right) T_1 + \left(-\frac{7}{120} F_{5,0} + \frac{1}{120} F_{5,1} + \frac{2}{27} \right) T_2.$$

Since $\{T_1, T_2\}$ must be free, we deduce:

$$0 = -\frac{1}{10} F_{5,0} + \frac{1}{120} F_{6,0} + \frac{1}{36} F_{5,1},$$

$$0 = -\frac{7}{120} F_{5,0} + \frac{1}{120} F_{5,1} + \frac{2}{27},$$

which we solve by assigning specific values to two Taylor coefficients of order 6:

$$F_{5,1} := 7 F_{5,0} - \frac{80}{9},$$

$$F_{6,0} := -\frac{34}{3} F_{5,0} + \frac{800}{27}.$$

Up to this point, $F_{5,0}$ is still free, and could be any real number $\eta \in \mathbb{R}$.

Next, from eqL at order 6:

$$F_{0,7} := 0, \quad F_{1,6} := 0, \quad F_{2,5} := 120, \quad F_{3,4} := 240, \quad F_{4,3} := 90,$$

$$F_{5,2} := 50 F_{5,0} - \frac{800}{9},$$

and it remains:

$$0 \stackrel{6,0}{=} \left(\frac{1}{720} F_{7,0} - \frac{40}{243} - \frac{7}{160} F_{5,0}^2 + \frac{8}{45} F_{5,0} \right) T_1 + \left(\frac{1}{720} F_{6,1} - \frac{22}{81} + \frac{67}{720} F_{5,0} \right) T_2,$$

$$0 \stackrel{5,1}{=} \left(\frac{47}{120} F_{5,0} - \frac{34}{27} + \frac{1}{120} F_{6,1} \right) T_1 + \left(-\frac{1}{20} F_{5,0} + \frac{1}{9} \right) T_2.$$

Surprisingly:

$$F_{5,0} := \frac{20}{9}.$$

Then:

$$F_{6,1} := \frac{140}{3}, \quad F_{7,0} := -\frac{280}{27}.$$

Assertion 1.13. *All higher order $F_{j,k}$ with $j + k \geq 8$ are uniquely determined as specific constants.* □

The infinitesimal symmetries are:

$$\begin{aligned} L = & \left([x - y - \frac{10}{9}u + 1] T_1 + [u - 2x] T_2 \right) \frac{\partial}{\partial x} \\ & + \left([\frac{10}{9}x - y - \frac{10}{9}u] T_1 + [-\frac{4}{3}x - y + \frac{8}{9}u + 1] T_2 \right) \frac{\partial}{\partial y} \\ & + \left([x + 2u] T_1 + [-3u] T_2 \right) \frac{\partial}{\partial u}. \end{aligned}$$

Proposition 1.14. *In the branch $F_{2,1} \neq 0$, $F_{3,1} \neq 0$, there is a single affinely homogeneous model:*

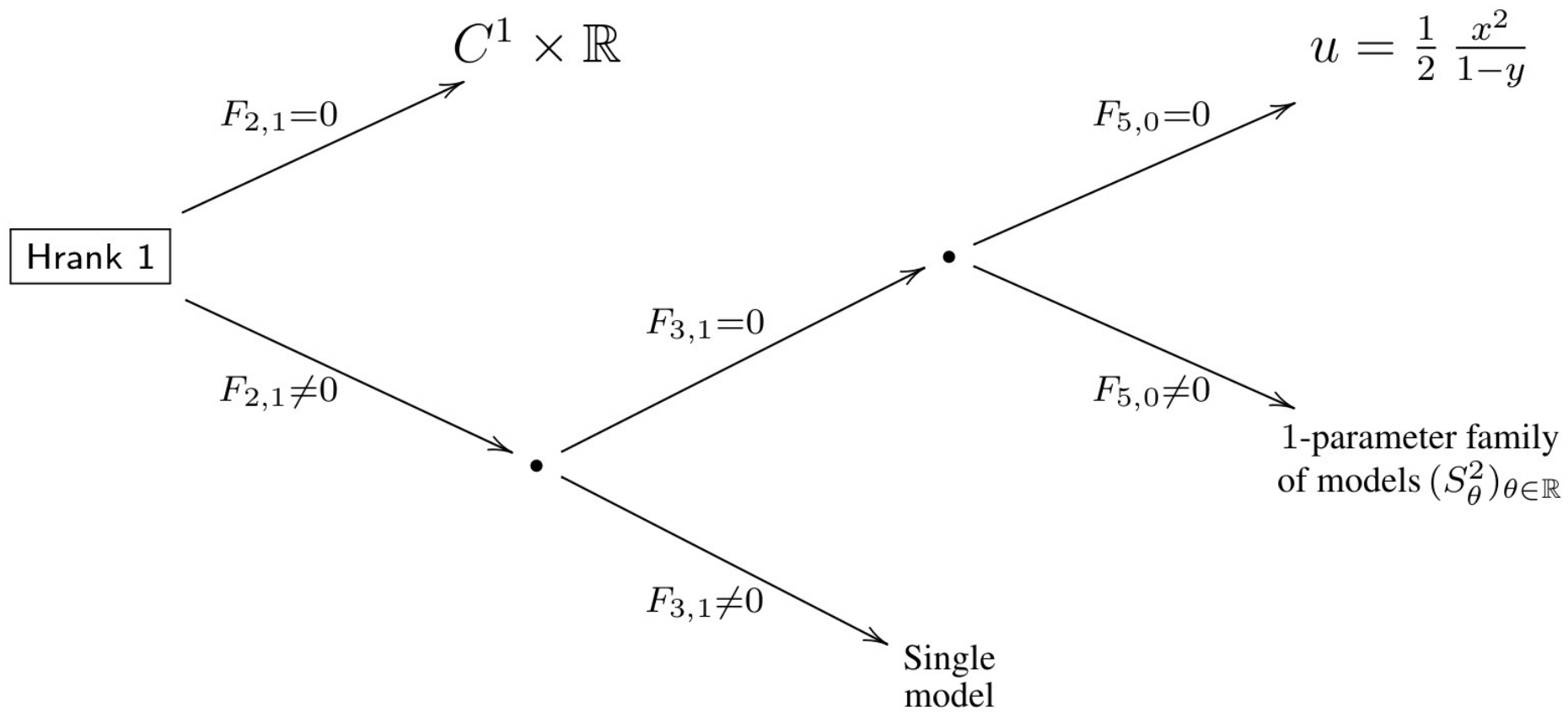
$$\begin{aligned}
u = & \frac{x^2}{2} \\
& + \frac{x^2 y}{2} \\
& + \frac{x^3 y}{6} + \frac{x^2 y^2}{2} \\
& + \frac{1}{54} x^5 + \frac{1}{2} x^3 y^2 + \frac{1}{2} x^2 y^3 \\
& + \frac{1}{162} x^6 + \frac{1}{18} x^5 y + \frac{1}{8} x^4 y^2 + x^3 y^3 + \frac{1}{2} x^2 y^4 \\
& - \frac{1}{486} x^7 + \frac{7}{108} x^6 y + \frac{5}{54} x^5 y^2 + \frac{5}{8} x^4 y^3 + \frac{5}{3} x^3 y^4 + \frac{1}{2} x^2 y^5 \\
& + \frac{5}{5832} x^8 + \frac{1}{162} x^7 y + \frac{1}{4} x^6 y^2 + \frac{47}{216} x^5 y^3 + \frac{15}{8} x^4 y^4 + \frac{5}{2} x^3 y^5 + \frac{1}{2} x^2 y^6 + O_{x,y}(9),
\end{aligned}$$

with 2-dimensional (simply-transitive) commutative affine Lie symmetry algebra: generated by:

$$\begin{aligned}
e_1 & := \left(x - y - \frac{10}{9} u + 1\right) \partial_x + \left(\frac{10}{9} x - y - \frac{10}{9} u\right) \partial_y + (x + 2u) \partial_u, \\
e_2 & := (u - 2x) \partial_x + \left(\frac{4}{3} x - y + \frac{8}{9} u + 1\right) \partial_y - 3u \partial_u,
\end{aligned}$$

having Lie bracket:

$$[e_1, e_2] = -e_1 - \frac{1}{3} e_2. \quad \square$$



2. Threefolds $H^3 \subset \mathbb{R}^4$

In \mathbb{R}^4 , consider an affine-linear map $(x, y, z, u) \mapsto (r, s, t, v)$ fixing the origin:

$$\begin{aligned} r &:= a_{1,1}x + a_{1,2}y + a_{1,3}z + b_1u, \\ s &:= a_{2,1}x + a_{2,2}y + a_{2,3}z + b_2u, \\ t &:= a_{3,1}x + a_{3,2}y + a_{3,3}z + b_3u, \\ v &:= c_1x + c_2y + c_3z + du, \end{aligned} \quad \text{with} \quad 0 \neq \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & b_1 \\ a_{2,1} & b_{2,2} & a_{2,3} & b_2 \\ a_{3,1} & b_{3,2} & a_{3,3} & b_3 \\ c_1 & c_2 & c_3 & d \end{vmatrix}.$$

Also, consider two graphed analytic hypersurfaces:

$$u = F(x, y, z) \quad (F(0,0,0) = 0) \quad \text{and} \quad v = G(r, s, t) \quad (0 = G(0,0,0)),$$

and assume that the above map is an affine equivalence $\{u = F\} \longrightarrow \{v = G\}$.

As in arxiv.org/abs/2202.03030/, the main hypothesis of constant Hessian rank 1, after elementary preliminary transformations:

$$u = \frac{x^2}{2} + O_{x,y,z}(3),$$

reads as:

$$1 \equiv \text{rank} \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix},$$

which is then equivalent to:

$$0 \equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xz} \\ F_{yx} & F_{yz} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{zx} & F_{zy} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xz} \\ F_{zx} & F_{zz} \end{vmatrix}.$$

By affine invariancy of the Hessian matrix rank, the same holds about $v = \frac{r^2}{2} + O_{r,s,t}(3)$.

The fundamental equation which holds identically in $\mathbb{R}\{x, y, z\}$:

$$0 \equiv \text{eqFG}(x, y, z),$$

writes:

$$\begin{aligned} \text{eqFG} &:= -c_1 x - c_2 y - c_3 z - d F(x, y, z) \\ &+ G \left(a_{1,1}x + a_{1,2}y + a_{1,3}z + b_1 F(x, y, z), \right. \\ &\quad a_{2,1}x + a_{2,2}y + a_{2,3}z + b_2 F(x, y, z), \\ &\quad \left. a_{3,1}x + a_{3,2}y + a_{3,3}z + b_3 F(x, y, z) \right). \end{aligned}$$

Also, an affine vector field:

$$\begin{aligned}
 L = & (T_1 + A_{1,1}x + A_{1,2}y + A_{1,3}z + B_1u) \frac{\partial}{\partial x} \\
 & + (T_2 + A_{2,1}x + A_{2,2}y + A_{2,3}z + B_2u) \frac{\partial}{\partial y} \\
 & + (T_3 + A_{3,1}x + A_{3,2}y + A_{3,3}z + B_3u) \frac{\partial}{\partial z} \\
 & + (T_0 + C_1x + C_2y + C_3z + Du) \frac{\partial}{\partial u},
 \end{aligned}$$

is tangent to $\{u = F(x, y, z)\}$ if and only if:

$$\begin{aligned}
 0 & \equiv \text{eqL}(x, y, z) \\
 & =: L(-u + F(x, y, z)) \Big|_{u=F(x,y,z)},
 \end{aligned}$$

identically as power series in $\mathbb{R}\{x, y, z\}$.

According to Theorems 1.4, 13.1, 1.5, 25.2 in arxiv.org/abs/2202.03030/, if $H^3 \subset \mathbb{R}^4$ is not affinely equivalent to a product with \mathbb{R}^1 or with \mathbb{R}^2 , its graphing function $F(x, y, z)$ can be *pre-normalized* — that is, *normalized before creating any branching* — up to order $3 + 5 = 8$ included

and modulo $O_{y,z}(3)$ as:

$$\begin{aligned}
u = & \frac{x^2}{2} \\
& + \frac{x^2y}{2} \\
& + \frac{x^3z}{6} + \frac{x^2y^2}{2} \\
& + F_{4,1,0} \frac{x^4y}{24} + \frac{x^3yz}{2} \\
& + F_{6,0,0} \frac{x^6}{720} + F_{5,1,0} \frac{x^5y}{120} + F_{4,1,0} \frac{x^4y^2}{6} + \frac{x^4z^2}{8} \\
& + F_{7,0,0} \frac{x^7}{5040} + F_{6,1,0} \frac{x^6y}{720} + F_{6,0,1} \frac{x^6z}{720} + F_{5,1,0} \frac{x^5y^2}{24} + F_{4,1,0} \frac{x^5yz}{12} \\
& + F_{8,0,0} \frac{x^8}{40320} + F_{7,1,0} \frac{x^7y}{5040} + F_{7,0,1} \frac{x^7z}{5040} + \left(\frac{1}{120} F_{6,1,0} - \frac{1}{48} F_{6,0,0} + \frac{1}{72} F_{4,1,0}^2 \right) x^6y^2 + \left(\frac{1}{48} F_{5,1,0} + \frac{1}{120} F_{6,0,1} \right) x^6yz \\
& + O_{y,z}(3) + O_{x,y,z}(9).
\end{aligned}$$

The same prenormalization holds for $v = G(r, s, t)$, of course.

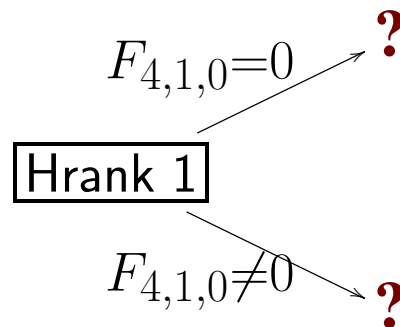
According to Section 25 of arxiv.org/abs/2202.03030/, already at order 6, the stability group is 1-dimensional:

$$\begin{bmatrix} a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{6}} \cdot$$

Therefore, $F_{4,1,0} \propto G_{4,1,0}$ is a relative invariant, the lowest order one in fact, and all other Taylor coefficients also are relative invariants, obviously. In fact:

$$0 \stackrel{\mathbf{4,1,0}}{=} -\frac{1}{24} F_{4,1,0} a_{1,1}^2 + \frac{1}{24} G_{4,1,0} a_{1,1}^4.$$

Consequently, we must open two branches:



Proposition 2.1. *In the branch $F_{4,1,0} \neq 0$, there are no affinely homogeneous models.*

Before starting the proof, without presenting the details, let us state up to order 6, that eqL gives the following value:

$$\begin{aligned}
L = & \left([1 - y - \frac{1}{2} F_{4,1,0} u] T_1 + u T_3 + x A_{1,1} \right) \frac{\partial}{\partial x} \\
& + \left([\frac{1}{2} F_{4,1,0} x - z + \frac{1}{5} F_{6,0,1} u - \frac{1}{5} F_{5,1,0} u] T_1 + [1 - y + \frac{1}{2} u F_{4,1,0}] T_2 - \frac{4}{3} x T_3 \right) \frac{\partial}{\partial y} \\
& + \left([-\frac{3}{10} F_{6,0,1} x + \frac{3}{10} F_{5,1,0} x - F_{4,1,0} y - \frac{1}{10} F_{6,0,0} u - \frac{1}{4} F_{4,1,0}^2 u] T_1 \right. \\
& \quad \left. + [-F_{4,1,0} x - 2z - \frac{1}{10} F_{5,1,0} u] T_2 + [1 - y + \frac{2}{3} F_{4,1,0} u] T_3 - z A_{1,1} \right) \frac{\partial}{\partial z} \\
& + \left(x T_1 + u T_2 + 2u A_{1,1} \right) \frac{\partial}{\partial u},
\end{aligned}$$

where T_1, T_2, T_3 and $A_{1,1}$ are free parameters.

Furthermore, at order 5, there remains 1 equation which behaves differently in the two branches:

$$0 \stackrel{\mathbf{4,1,0}}{=} \frac{1}{24} F_{5,1,0} T_1 + \frac{1}{8} F_{4,1,0} T_2 + \frac{1}{12} F_{4,1,0} A_{1,1},$$

since $A_{1,1}$ may be solved only if $0 \neq F_{4,1,0}$, and there remain 2 equations at order 6:

$$0 \stackrel{6,0,0}{=} \left(\frac{1}{720} F_{7,0,0} + \frac{1}{240} F_{4,1,0} F_{6,0,1} \right) T_1 + \left(\frac{1}{720} F_{6,1,0} - \frac{1}{720} F_{6,0,0} + \frac{1}{96} F_{4,1,0}^2 \right) T_2 \\ + \left(\frac{1}{720} F_{6,0,1} - \frac{1}{90} F_{5,1,0} \right) T_3 + \frac{1}{180} F_{6,0,0} A_{1,1},$$

$$0 \stackrel{5,1,0}{=} \left(\frac{1}{120} F_{6,1,0} - \frac{1}{24} F_{6,0,0} + \frac{1}{48} F_{4,1,0}^2 \right) T_1 + \frac{1}{30} F_{5,1,0} T_2 - \frac{1}{72} F_{4,1,0} T_3 + \frac{1}{40} F_{5,1,0} A_{1,1}.$$

Proof. If $F_{4,1,0} \neq 0$, looking at $\stackrel{4,1,0}{=}$ of eqFG above, we can normalize:

$$G_{4,1,0} = \pm 1,$$

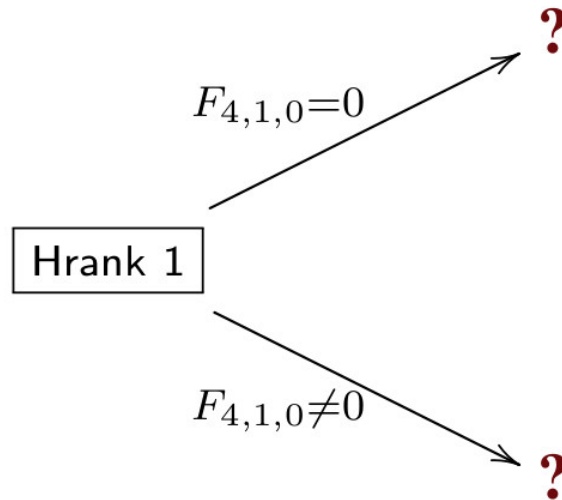
and symmetrically $F_{4,1,0} = \pm 1$. Then at the infinitesimal level, we may solve from $\stackrel{4,1,0}{=}$ of eqL:

$$A_{1,1} = \mp \frac{1}{2} F_{5,1,0} T_1 - \frac{3}{2} T_2,$$

whence by replacement in $\stackrel{5,1,0}{=}$ of eqL:

$$0 \stackrel{5,1,0}{=} \left(\frac{1}{120} F_{6,1,0} - \frac{1}{24} F_{6,0,0} \mp \frac{1}{80} F_{5,1,0}^2 + \frac{1}{48} \right) T_1 - \frac{1}{240} F_{5,1,0} T_2 \mp \frac{1}{72} T_3.$$

This always is a contradictory nontrivial linear relation between the transitivity parameters $\{T_1, T_2, T_3\}$, because $\mp \frac{1}{72} \neq 0$. \square

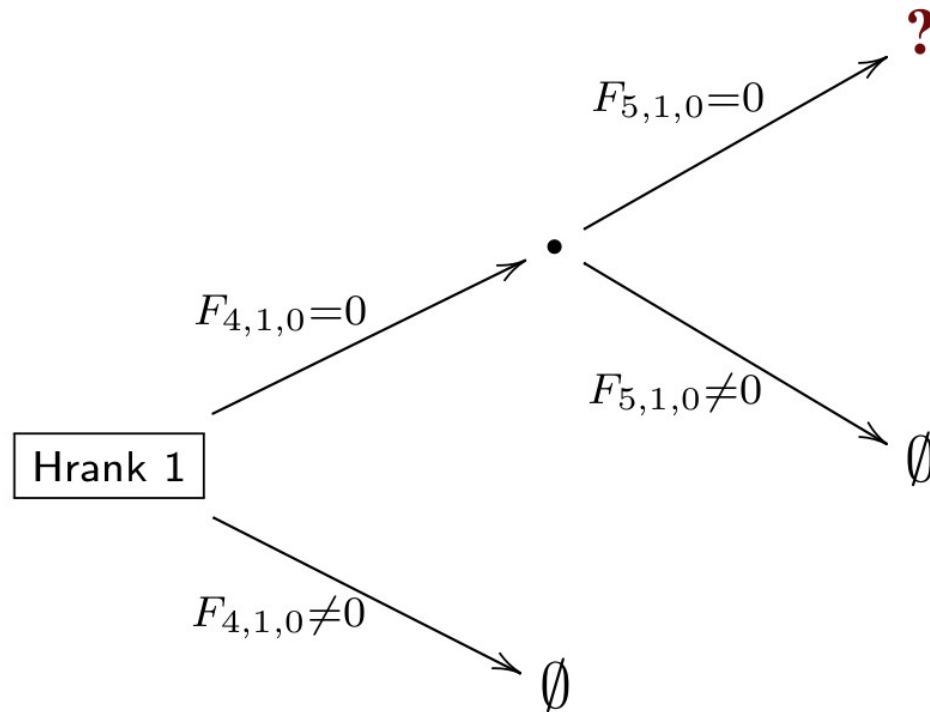


Therefore, $F_{4,1,0} = 0$ necessarily. At order 6, eqL consists of 2 equations:

$$0 \stackrel{\mathbf{6,0,0}}{=} -\frac{1}{720} F_{6,0,0} a_{1,1}^2 + \frac{1}{720} G_{6,0,0} a_{1,1}^6,$$

$$0 \stackrel{\mathbf{5,1,0}}{=} -\frac{1}{120} F_{5,1,0} a_{1,1}^2 + \frac{1}{120} G_{5,1,0} a_{1,1}^5.$$

Again, we must open two branches.

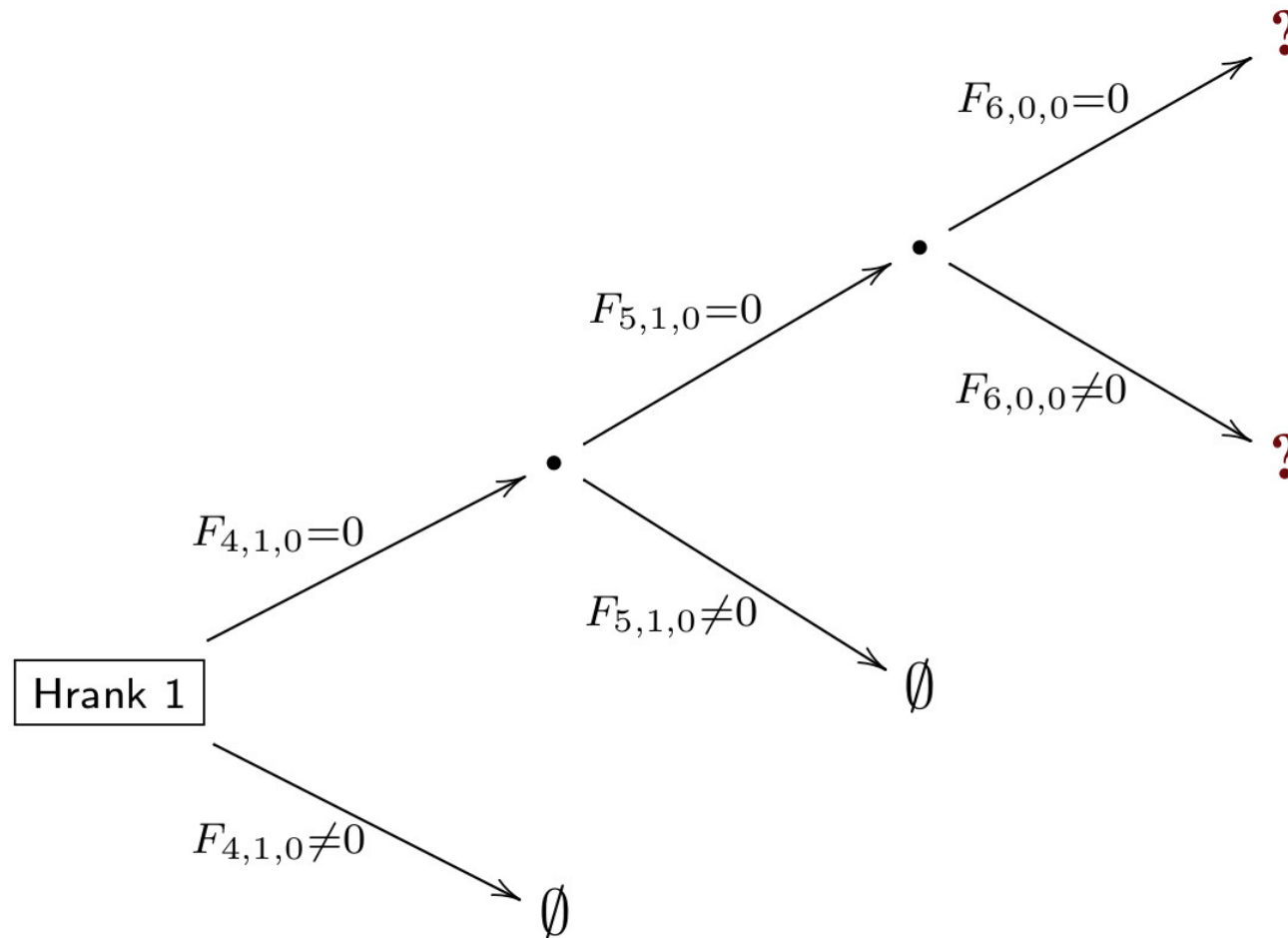


But quickly, $\overset{4,1,0}{\equiv}$ of eqL above

$$0 \overset{4,1,0}{\equiv} \frac{1}{24} F_{5,1,0} T_1 + 0 + 0,$$

forces $F_{5,1,0} = 0$, so that one branch is void.

Similarly, the relative invariancy of $F_{6,0,0}$ creates two branches:



Since $\overset{5,1,0}{=} 0$ of eqL becomes:

$$0 \overset{5,1,0}{=} \left(\frac{1}{120} F_{6,1,0} - \frac{1}{24} F_{6,0,0} + 0 \right) T_1 + 0 + 0 + 0,$$

we have:

$$F_{6,1,0} = 5 F_{6,0,0}.$$

Then at orders 6 and 7, eqL consists of:

$$\begin{aligned}
0 \stackrel{6,0,0}{=} & \frac{1}{720} F_{7,0,0} T_1 + \frac{1}{180} F_{6,0,0} T_2 + \frac{1}{720} F_{6,0,1} T_3 + \frac{1}{180} F_{6,0,0} A_{1,1}, \\
0 \stackrel{7,0,0}{=} & \left(\frac{1}{5040} F_{8,0,0} - \frac{1}{2400} F_{6,0,1}^2 \right) T_1 + \left(-\frac{1}{5040} F_{7,0,0} + \frac{1}{5040} F_{7,1,0} \right) T_2 \\
& + \left(-\frac{1}{270} F_{6,0,0} + \frac{1}{5040} F_{7,0,1} \right) T_3 + \frac{1}{1008} F_{7,0,0} A_{1,1}, \\
0 \stackrel{6,1,0}{=} & \left(-\frac{1}{720} F_{7,0,0} + \frac{1}{720} F_{7,1,0} \right) T_1 + \frac{1}{36} F_{6,0,0} T_2 + \frac{1}{144} F_{6,0,1} T_3 + \frac{1}{36} F_{6,0,0} A_{1,1}, \\
0 \stackrel{6,0,1}{=} & \left(\frac{1}{720} F_{7,0,1} - \frac{1}{45} F_{6,0,0} \right) T_1 + \frac{1}{240} F_{6,0,1} T_2 + \frac{1}{240} F_{6,0,1} A_{1,1}.
\end{aligned}$$

Proposition 2.2. *In the branch $F_{6,0,0} \neq 0$, there are no affinely homogeneous models.*

Proof. From $\stackrel{6,0,0}{=}$ of eqFG, we may normalize:

$$G_{6,0,0} = \pm 1 = F_{6,0,0},$$

then from $\stackrel{6,0,0}{=}$ of eqL:

$$A_{1,1} := \mp \frac{1}{4} F_{7,0,0} T_1 - T_2 \mp \frac{1}{4} F_{6,0,1} T_3.$$

A replacement gives:

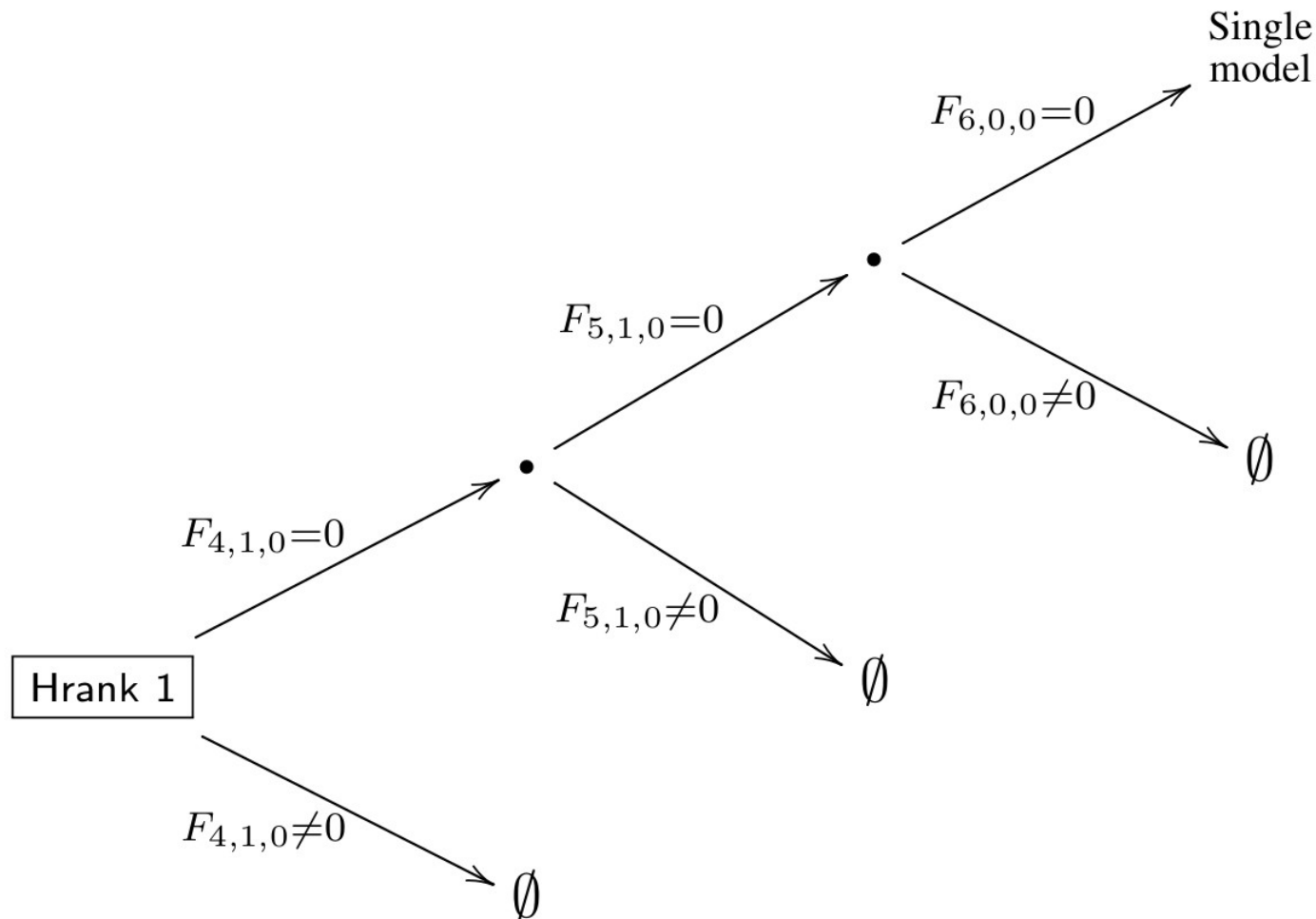
$$0 \stackrel{6,0,1}{=} \mp \frac{1}{960} F_{6,0,1}^2 T_3 + \left(\frac{1}{720} F_{7,0,1} \mp \frac{1}{960} F_{7,0,0} F_{6,0,1} \mp \frac{1}{45} \right) T_1,$$

whence $F_{6,0,1} = 0$ necessarily, and then:

$$0 \stackrel{\mathbf{7,0,0}}{=} * T_1 + * T_2 + \left(\frac{1}{5040} F_{7,0,1} \mp \frac{1}{270} \right) T_3,$$

$$0 \stackrel{\mathbf{6,0,1}}{=} 0 + \left(\frac{1}{720} F_{7,0,1} \mp \frac{1}{45} \right) T_1,$$

where $*$ are unimportant, but this gives the two noncoinciding values $\pm \frac{56}{3}$ and ± 16 for $F_{7,0,1}$. \square



Theorem 2.3. *Among constant Hessian rank 1 hypersurfaces $H^3 \subset \mathbb{R}^4$, there is a single affinely homogeneous model, lying in the branch $F_{2,1} \neq 0$, $F_{3,1} = 0$, $F_{5,0} = 0$, of equation:*

$$\begin{aligned}
u = & \frac{x^2}{2} \\
& + \frac{x^2 y}{2} \\
& + \frac{x^3 z}{6} + \frac{x^2 y^2}{2} \\
& + \frac{x^3 y z}{2} + \frac{x^2 y^3}{2} \\
& + \frac{1}{8} x^4 z^2 + x^3 y^2 z + \frac{1}{2} x^2 y^4 \\
& + \frac{5}{8} x^4 y z^2 + \frac{5}{3} x^3 y^3 z + \frac{1}{2} x^2 y^5, \\
& + \frac{1}{8} x^5 z^3 + \frac{15}{8} x^4 y^2 z^2 + \frac{5}{2} x^3 y^4 z + \frac{1}{2} x^2 y^6 \\
& + \frac{7}{8} x^5 y z^3 + \frac{35}{8} x^4 y^3 z^2 + \frac{7}{2} x^3 y^5 z + \frac{1}{2} x^2 y^7 \\
& + \frac{7}{48} x^6 z^4 + \frac{7}{2} x^5 y^2 z^3 + \frac{35}{4} x^4 y^4 z^2 + \frac{14}{3} x^3 y^6 z + \frac{1}{2} x^2 y^8 + \\
& + O_{x,y,z}(11),
\end{aligned}$$

with 4-dimensional affine symmetry algebra generated by:

$$\begin{aligned} e_1 &:= (1 - y) \partial_x - z \partial_y + x \partial_u, \\ e_2 &:= (1 - y) \partial_y - 2z \partial_z + u \partial_u, \\ e_3 &:= u \partial_x - \frac{4}{3} x \partial_y + (1 - y) \partial_z, \\ e_4 &:= x \partial_x - z \partial_z + 2u \partial_u. \end{aligned}$$

Proof. Putting $F_{6,0,0} := 0$, and knowing $F_{6,1,0} = 5 F_{6,0,0} = 0$, at order 6 for eqL, it remains only:

$$0 \stackrel{\mathbf{6,0,0}}{=} \frac{1}{720} F_{7,0,0} T_1 + \frac{1}{720} F_{6,0,1} T_3,$$

whence $0 = F_{7,0,0} = F_{6,0,1}$.

At order 7, eqL reads:

$$\begin{aligned} 0 &\stackrel{\mathbf{7,0,0}}{=} \frac{1}{5040} F_{8,0,0} T_1 + \frac{1}{5040} F_{7,1,0} T_2 + \frac{1}{5040} F_{7,0,1} T_3, \\ 0 &\stackrel{\mathbf{6,1,0}}{=} \frac{1}{720} F_{7,1,0} T_1, \\ 0 &\stackrel{\mathbf{6,0,1}}{=} \frac{1}{720} F_{7,0,1} T_1, \end{aligned}$$

whence $F_{8,0,0} = F_{7,1,0} = F_{7,0,1}$.

Generally, one can see that for all $\mu \geq 7$:

$$0 = F_{\mu,0,0} = F_{\mu-1,1,0} = F_{\mu,0,1}. \quad \square$$

Corollary 2.4. *A closed expression for the graphing function $F(x, y, z)$ is:*

$$u = \frac{1}{3z^2} \left\{ (1 - 2y + y^2 - 2xz)^{3/2} - (1 - y)(1 - 2y + y^2 - 3xz) \right\}.$$

Proof. By expanding the numerator in power series, one realizes that the singularity $\frac{1}{z^2}$ is removable, and that the power series expansion matches with that of Theorem 2.3 up to order 10 monomials.

On the other hand, one verifies that e_1, e_2, e_3, e_4 are infinitesimal symmetries of this closed form. \square

3. Fourfolds $H^4 \subset \mathbb{R}^5$

In \mathbb{R}^5 , consider an affine-linear map $(x, y, z, w, u) \mapsto (r, s, t, p, v)$ fixing the origin:

$$r := a_{1,1}x + a_{1,2}y + a_{1,3}z + a_{1,4}w + b_1u,$$

$$s := a_{2,1}x + a_{2,2}y + a_{2,3}z + a_{2,4}w + b_2u,$$

$$t := a_{3,1}x + a_{3,2}y + a_{3,3}z + a_{3,4}w + b_3u,$$

$$p := a_{4,1}x + a_{4,2}y + a_{4,3}z + a_{4,4}w + b_4u,$$

$$v := c_1x + c_2y + c_3z + c_4w + du,$$

with:

$$0 \neq \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & b_1 \\ a_{2,1} & b_{2,2} & a_{2,3} & a_{2,4} & b_2 \\ a_{3,1} & b_{3,2} & a_{3,3} & a_{3,4} & b_3 \\ a_{4,1} & b_{4,2} & a_{4,3} & a_{4,4} & b_4 \\ c_1 & c_2 & c_3 & c_4 & d \end{vmatrix}.$$

Also, consider two graphed analytic hypersurfaces:

$$u = F(x, y, z, w) \quad (F(0,0,0,0) = 0),$$

$$v = G(r, s, t, p) \quad (G(0,0,0,0) = 0),$$

and assume that the above map is an affine equivalence $\{u = F\} \longrightarrow \{v = G\}$.

The main hypothesis of constant Hessian rank 1, after elementary preliminary transformations:

$$u = \frac{x^2}{2} + O_{x,y,z,w}(3),$$

reads as:

$$1 \equiv \text{rank} \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} & F_{xw} \\ F_{yx} & F_{yy} & F_{yz} & F_{yw} \\ F_{zx} & F_{zy} & F_{zz} & F_{zw} \\ F_{wx} & F_{wy} & F_{wz} & F_{ww} \end{bmatrix},$$

which is then equivalent to:

$$\begin{aligned} 0 &\equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xz} \\ F_{yx} & F_{yz} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xw} \\ F_{yx} & F_{yw} \end{vmatrix} \\ &\equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{zx} & F_{zy} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xz} \\ F_{zx} & F_{zz} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xw} \\ F_{zx} & F_{zw} \end{vmatrix}, \\ &\equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{wx} & F_{wy} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xz} \\ F_{wx} & F_{wz} \end{vmatrix} \equiv \begin{vmatrix} F_{xx} & F_{xw} \\ F_{wx} & F_{ww} \end{vmatrix}. \end{aligned}$$

By affine invariancy of the Hessian matrix rank, the same holds about $v = \frac{r^2}{2} + O_{r,s,t}(3)$.

The fundamental equation which holds identically in $\mathbb{R}\{x, y, z, w\}$:

$$0 \equiv \text{eqFG}(x, y, z, w),$$

writes:

$$\begin{aligned} \text{eqFG} &:= -c_1 x - c_2 y - c_3 z - c_4 w - d F(x, y, z) \\ &+ G\left(a_{1,1}x + a_{1,2}y + a_{1,3}z + a_{1,4}w + b_1 F(x, y, z, w), \right. \\ &\quad a_{2,1}x + a_{2,2}y + a_{2,3}z + a_{2,4}w + b_2 F(x, y, z, w), \\ &\quad a_{3,1}x + a_{3,2}y + a_{3,3}z + a_{3,4}w + b_3 F(x, y, z, w), \\ &\quad \left. a_{4,1}x + a_{4,2}y + a_{4,3}z + a_{4,4}w + b_4 F(x, y, z, w)\right). \end{aligned}$$

Also, an affine vector field:

$$\begin{aligned}
L = & (T_1 + A_{1,1}x + A_{1,2}y + A_{1,3}z + A_{1,4}w + B_1u) \frac{\partial}{\partial x} \\
& + (T_2 + A_{2,1}x + A_{2,2}y + A_{2,3}z + A_{2,4}w + B_2u) \frac{\partial}{\partial y} \\
& + (T_3 + A_{3,1}x + A_{3,2}y + A_{3,3}z + A_{3,4}w + B_3u) \frac{\partial}{\partial z} \\
& + (T_4 + A_{4,1}x + A_{4,2}y + A_{4,3}z + A_{4,4}w + B_4u) \frac{\partial}{\partial w} \\
& + (T_0 + C_1x + C_2y + C_3z + C_4w + Du) \frac{\partial}{\partial u},
\end{aligned}$$

is tangent to $\{u = F(x, y, z, w)\}$ if and only if:

$$0 \equiv \text{eqL}(x, y, z, w)$$

$$=: L(-u + F(x, y, z, w)) \Big|_{u=F(x,y,z,w)},$$

identically as power series in $\mathbb{R}\{x, y, z, w\}$.

According to Theorems 1.4, 13.1, 1.5, 25.2 in arxiv.org/abs/2202.03030/, if $H^4 \subset \mathbb{R}^5$ is not affinely equivalent to a product with \mathbb{R}^1 or \mathbb{R}^2 or \mathbb{R}^3 , its graphing function $F(x, y, z, w)$ can be *pre-normalized* — that is,

normalized before creating any branching — up to order $4 + 5 = 9$ included and modulo $O_{y,z,w}(3)$ as:

$$\begin{aligned}
u = & \frac{x^2}{2} \\
& + \frac{x^2 y}{2} \\
& + \frac{x^3 z}{6} + \frac{x^2 y^2}{2} \\
& + \frac{x^4 w}{24} + \frac{x^3 y z}{2} + \frac{x^2 y^3}{2} \\
& + F_{5,1,0,0} \frac{x^5 y}{120} + \frac{x^4 y w}{6} + \frac{x^4 z^2}{8} + x^3 y^2 z \\
& + F_{7,0,0,0} \frac{x^7}{5040} + F_{6,1,0,0} \frac{x^6 y}{720} + F_{6,0,0,1} \frac{x^6 w}{720} + F_{5,1,0,0} \frac{x^5 y^2}{24} + \frac{x^5 z w}{12} \\
& + \frac{1}{40320} F_{8,0,0,0} x^8 + \frac{1}{5040} F_{7,1,0,0} x^7 y + \frac{1}{5040} F_{7,0,1,0} x^7 z + \frac{1}{5040} F_{7,0,0,1} x^7 w \\
& \quad + \frac{1}{120} F_{6,1,0,0} x^6 y^2 + \frac{1}{48} F_{5,1,0,0} x^6 y z + \frac{1}{120} F_{6,0,0,1} x^6 y w + \frac{1}{72} x^6 w^2 \\
& + \frac{1}{362880} F_{9,0,0,0} x^9 + \frac{1}{40320} F_{8,1,0,0} x^8 y + \frac{1}{40320} F_{8,0,1,0} x^8 z + \frac{1}{40320} F_{8,0,0,1} x^8 w \\
& \quad + \frac{1}{10080} (14 F_{7,1,0,0} - 42 F_{7,0,0,0}) x^7 y^2 + \frac{1}{5040} (7 F_{7,0,1,0} + 21 F_{6,1,0,0}) x^7 y z \\
& \quad + \frac{1}{5040} (7 F_{7,0,0,1} + 35 F_{5,1,0,0}) x^7 y w + \frac{1}{240} F_{6,0,0,1} x^7 z w \\
& + O_{y,z,w}(3) + O_{x,y,z,w}(10).
\end{aligned}$$

According to Section 25 of arxiv.org/abs/2202.03030/, already at order 7, the stability group is 1-dimensional:

$$\begin{bmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{1,1}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a_{1,1}^2} & 0 \\ 0 & 0 & 0 & 0 & a_{1,1}^2 \end{bmatrix}^7 \cdot$$

Moreover, $F_{5,1,0,0} \propto G_{5,1,0,0}$ is a relative invariant, the lowest order one in fact, and all other Taylor coefficients also are relative invariants, obviously. In fact:

$$0 \stackrel{5100}{=} -\frac{1}{120} F_{5,1,0,0} a_{1,1}^2 + \frac{1}{120} G_{5,1,0,0} a_{1,1}^5.$$

For the moment, we do not open a branching here.

Up to order 6, eqL gives:

$$\begin{aligned}
L = & \left(T_1 + A_{1,1} x - T_1 y + \left[\frac{1}{5} F_{6,0,0,1} T_1 + \frac{2}{3} T_3 \right] u \right) \frac{\partial}{\partial x} \\
& + \left(T_2 + \left[-T_3 - \frac{1}{5} F_{6,0,0,1} T_1 \right] x - T_2 y - T_1 z + \left[\frac{1}{2} T_4 - \frac{1}{5} F_{5,1,0,0} T_1 \right] u \right) \frac{\partial}{\partial y} \\
& + \left(T_3 + \left[\frac{3}{10} F_{5,1,0,0} T_1 - T_4 \right] x - T_3 y + \left[-2T_2 - A_{1,1} \right] z - T_1 w \right. \\
& \quad \left. + \left[\left(\frac{1}{10} F_{7,0,1,0} - \frac{1}{10} F_{6,1,0,0} + \frac{1}{25} F_{6,0,0,1}^2 \right) T_1 + \frac{3}{10} F_{5,1,0,0} T_2 - \frac{1}{15} F_{6,0,0,1} T_3 \right] u \right) \frac{\partial}{\partial z} \\
& + \left(T_4 + \left[\left(\frac{1}{5} F_{6,1,0,0} - \frac{1}{5} F_{7,0,1,0} - \frac{2}{25} F_{6,0,0,1}^2 \right) T_1 - \frac{4}{5} F_{5,1,0,0} T_2 + \frac{2}{15} F_{6,0,0,1} T_3 \right] x + \left[-F_{5,1,0,0} T_1 - T_4 \right] y \right. \\
& \quad \left. + \left[\frac{2}{5} F_{6,0,0,1} T_1 - \frac{2}{3} T_3 \right] z + \left[-3T_2 - 2A_{1,1} \right] w \right. \\
& \quad \left. + \left[\left(\frac{2}{25} F_{5,1,0,0} F_{6,0,0,1} - \frac{1}{15} F_{7,0,0,0} \right) T_1 - \frac{1}{15} F_{6,1,0,0} T_2 + \frac{2}{5} F_{5,1,0,0} T_3 - \frac{1}{15} F_{6,0,0,1} T_4 \right] u \right) \frac{\partial}{\partial w} \\
& + \left(T_1 x + \left[T_2 + 2A_{1,1} \right] u \right) \frac{\partial}{\partial u},
\end{aligned}$$

with the four transitivity parameters T_1, T_2, T_3, T_4 , plus a single possible isotropy parameter $A_{1,1}$.

Up to order 8, the remaining equations of eqL are:

$$0 \stackrel{\mathbf{5100}}{=} \frac{1}{120} F_{6,1,0,0} T_1 + \frac{1}{30} F_{5,1,0,0} T_2 + \frac{1}{40} F_{5,1,0,0} A_{1,1},$$

$$\begin{aligned}
0 &\stackrel{7000}{=} \left(\frac{1}{5040} F_{8,0,0,0} - \frac{1}{3600} F_{7,0,1,0} F_{6,0,0,1} - \frac{1}{9000} F_{6,0,0,1}^3 - \frac{1}{1200} F_{5,1,0,0}^2 \right) T_1 \\
&+ \left(\frac{1}{5040} F_{7,1,0,0} - \frac{1}{5040} F_{7,0,0,0} - \frac{1}{900} F_{6,0,0,1} F_{5,1,0,0} \right) T_2 \\
&+ \left(\frac{1}{5040} F_{7,0,1,0} - \frac{1}{720} F_{6,1,0,0} + \frac{1}{5040} F_{6,0,0,1}^2 \right) T_3 + \left(\frac{1}{5040} F_{7,0,0,1} + \frac{1}{480} F_{5,1,0,0} \right) T_4 + \frac{1}{1008} F_{7,0,0,0} A_{1,1}, \\
0 &\stackrel{6100}{=} \left(\frac{1}{720} F_{7,1,0,0} - \frac{1}{120} F_{7,0,0,0} - \frac{7}{1800} F_{5,1,0,0} F_{6,0,0,1} \right) T_1 + \frac{1}{144} F_{6,1,0,0} T_2 - \frac{1}{720} F_{5,1,0,0} T_3 + \frac{1}{180} F_{6,1,0,0} A_{1,1} \\
&0 \stackrel{6001}{=} \left(\frac{1}{720} F_{7,0,0,1} + \frac{1}{240} F_{5,1,0,0} \right) T_1 + \frac{1}{360} F_{6,0,0,1} T_2 - \boxed{\frac{1}{288} T_4} + \frac{1}{360} F_{6,0,0,1} A_{1,1}, \\
0 &\stackrel{8000}{=} \left(\frac{1}{40320} F_{9,0,0,0} - \frac{1}{25200} F_{7,0,1,0} F_{7,0,0,1} - \frac{1}{25200} F_{7,1,0,0} F_{6,0,0,1} + \frac{1}{16800} F_{7,0,1,0} F_{5,1,0,0} + \frac{1}{25200} F_{7,0,0,1} F_{6,1,0,0} \right. \\
&\quad \left. - \frac{1}{63000} F_{7,0,0,1} F_{6,0,0,1}^2 + \frac{1}{7560} F_{7,0,0,0} F_{6,0,0,1} + \frac{1}{18000} F_{5,1,0,0} F_{6,0,0,1}^2 - \frac{1}{7200} F_{5,1,0,0} F_{6,1,0,0} \right) T_1 \\
&+ \left(\frac{1}{40320} F_{8,1,0,0} - \frac{1}{40320} F_{8,0,0,0} - \frac{1}{6300} F_{5,1,0,0} F_{7,0,0,1} - \frac{1}{21600} F_{6,1,0,0} F_{6,0,0,1} \right) T_2 \\
&+ \left(\frac{1}{40320} F_{8,0,1,0} - \frac{1}{5040} F_{7,1,0,0} + \frac{1}{1680} F_{7,0,0,0} + \frac{1}{37800} F_{6,0,0,1} F_{7,0,0,1} + \frac{1}{3600} F_{5,1,0,0} F_{6,0,0,1} \right) T_3 \\
&+ \left(\frac{1}{40320} F_{8,0,0,1} - \frac{1}{5040} F_{7,0,1,0} + \frac{1}{2880} F_{6,1,0,0} - \frac{1}{21600} F_{6,0,0,1}^2 \right) T_4 + \frac{1}{6720} F_{8,0,0,0} A_{1,1}, \\
0 &\stackrel{7100}{=} \left(\frac{1}{5040} F_{8,1,0,0} - \frac{1}{5040} F_{8,0,0,0} - \frac{1}{5040} F_{5,1,0,0} F_{7,0,0,1} - \frac{1}{600} F_{7,0,1,0} F_{6,0,0,1} \right. \\
&\quad \left. - \frac{1}{1800} F_{6,1,0,0} F_{6,0,0,1} - \frac{3}{800} F_{5,1,0,0}^2 - \frac{1}{1500} F_{6,0,0,1}^3 \right) T_1 + \\
&+ \left(\frac{1}{420} F_{7,1,0,0} - \frac{1}{120} F_{7,0,0,0} - \frac{1}{150} F_{5,1,0,0} F_{6,0,0,1} \right) T_2 + \left(\frac{1}{840} F_{7,0,1,0} - \frac{19}{2160} F_{6,1,0,0} + \frac{1}{900} F_{6,0,0,1}^2 \right) T_3 \\
&+ \left(\frac{1}{840} F_{7,0,0,1} + \frac{1}{90} F_{5,1,0,0} \right) T_4 + \frac{1}{1008} F_{7,1,0,0} A_{1,1},
\end{aligned}$$

$$\begin{aligned}
0 \stackrel{7010}{=} & \left(\frac{1}{5040} F_{8,0,1,0} - \frac{1}{5040} F_{7,1,0,0} - \frac{7}{2160} F_{7,0,0,0} + \frac{1}{12600} F_{6,0,0,1} F_{7,0,0,1} - \frac{1}{3600} F_{5,1,0,0} F_{6,0,0,1} \right) T_1 \\
& + \left(\frac{1}{1260} F_{7,0,1,0} + \frac{1}{1080} F_{6,1,0,0} \right) T_2 + \left(-\frac{1}{7560} F_{7,0,0,1} - \frac{1}{720} F_{5,1,0,0} \right) T_3 + \frac{1}{1080} F_{6,0,0,1} T_4 + \frac{1}{1260} F_{7,0,1,0} A_{1,1}, \\
0 \stackrel{7001}{=} & \left(\frac{1}{5040} F_{8,0,0,1} - \frac{1}{1120} F_{7,0,1,0} + \frac{1}{1440} F_{6,1,0,0} - \frac{1}{1200} F_{6,0,0,1}^2 \right) T_1 \\
& + \left(\frac{1}{1680} F_{7,0,0,1} - \frac{1}{1440} F_{5,1,0,0} \right) T_2 + \frac{1}{1680} F_{7,0,0,1} A_{1,1}.
\end{aligned}$$

Observation 3.1. $F_{6,0,0,1} \neq 0$, necessarily.

Proof. If we would have $F_{6,0,0,1} = 0$, because of the presence of $-\frac{1}{288} T_4$, the equation $\stackrel{6001}{=}$ above would be a nontrivial linear dependence relation between the transitivity parameters T_1, T_2, T_3, T_4 , which is forbidden. \square

From eqFG:

$$0 \stackrel{6001}{=} -\frac{1}{720} F_{6,0,0,1} a_{1,1}^2 + \frac{1}{720} G_{6,0,0,1} a_{1,1}^4,$$

we see that we can normalize:

$$G_{6,0,0,1} := 1 \quad \text{or} \quad G_{6,0,0,1} := -1,$$

and the same about $F_{6,0,0,1}$. Stabilization of this last normalization requires $a_{1,1} := 1$, and if there exists any homogeneous model, it can only be simply transitive.

As a first case, put $F_{6,0,0,1} := 1$ everywhere, solve from $\overset{6001}{=}$:

$$A_{1,1} := \left(-\frac{3}{2} F_{5,1,0,0} - \frac{1}{2} F_{7,0,0,1} T_1 \right) T_1 - T_2 + \frac{5}{4} T_4,$$

and replace this value of $A_{1,1}$ everywhere. Then $\overset{5100}{=}$ becomes:

$$\begin{aligned} 0 \overset{5100}{=} & \left(\frac{1}{120} F_{6,1,0,0} - \frac{1}{80} F_{5,1,0,0} F_{7,0,0,1} - \frac{3}{80} F_{5,1,0,0}^2 \right) T_1 \\ & + \frac{1}{120} F_{5,1,0,0} T_2 + \frac{1}{32} F_{5,1,0,0} T_4, \end{aligned}$$

whence necessarily:

$$F_{5,1,0,0} = 0$$

and then:

$$F_{6,1,0,0} = 0.$$

This necessary vanishing $F_{5,1,0,0} = 0$ *a posteriori* explains why we did not open a branch *supra*.

Therefore, put $F_{5,1,0,0} := 0$ and $F_{6,1,0,0} := 0$ everywhere. Then $\stackrel{7000}{=}$ becomes:

$$0 \stackrel{7000}{=} \left(\frac{1}{5040} F_{8,0,0,0} - \frac{1}{3600} F_{7,0,1,0} - \frac{1}{2016} F_{7,0,0,0} F_{7,0,0,1} - \frac{1}{1900} \right) T_1 \\ + \left(\frac{1}{5040} F_{7,1,0,0} - \frac{1}{840} F_{7,0,0,0} \right) T_2 + \left(\frac{1}{5040} F_{7,0,1,0} + \frac{1}{5400} \right) T_3 \\ + \left(\frac{1}{5040} F_{7,0,0,1} + \frac{5}{4032} F_{7,0,0,0} \right) T_4.$$

It follows:

$$F_{7,0,1,0} = -\frac{5040}{5400} = -\frac{14}{15},$$

whence:

$$0 \stackrel{7001}{=} \left(\frac{1}{5040} F_{8,0,0,1} - \frac{1}{3360} F_{7,0,0,1}^2 \right) T_1 + \frac{1}{1344} F_{7,0,0,1} T_4,$$

so that:

$$F_{7,0,0,1} = 0, \quad F_{8,0,0,1} = 0.$$

Then:

$$0 \stackrel{7000}{=} \left(\frac{1}{5040} F_{8,0,0,0} + \frac{1}{6750} \right) T_1 + \left(\frac{1}{5040} F_{7,1,0,0} - \frac{1}{840} F_{7,0,0,0} \right) T_2 \\ + \frac{5}{4032} F_{7,0,0,0} T_4.$$

Thus:

$$\begin{aligned}
 F_{7,0,0,0} &= 0, & F_{7,1,0,0} &= 0, & F_{8,0,0,0} &= -\frac{56}{75}, \\
 F_{7,1,0,0} &= 0, & F_{8,0,1,0} &= 0, & F_{8,1,0,0} &= -\frac{392}{75}, & F_{9,0,0,0} &= 0.
 \end{aligned}$$

The second case $F_{6,0,0,1} = -1$ is treated similarly.

Theorem 3.2. *Among constant Hessian rank 1 hypersurfaces $H^4 \subset \mathbb{R}^5$, there are only two affinely homogeneous models, of equations depending on some sign choices \pm or \mp :*

$$\begin{aligned}
 u &= \frac{x^2}{2} \\
 &+ \frac{x^2y}{2} \\
 &+ \frac{x^3z}{6} + \frac{x^2y^2}{2} \\
 &+ \frac{x^4w}{24} + \frac{x^3yz}{2} + \frac{x^2y^3}{2} \\
 &+ \frac{x^4yw}{6} + \frac{x^4z^2}{8} + x^3y^2z + \frac{x^2y^4}{2} \\
 &\pm \frac{x^6w}{720} + \frac{1}{12}x^5zw + \frac{5}{12}x^4y^2w + \frac{5}{8}x^4yz^2 + \frac{5}{3}x^3y^3z + \frac{1}{2}x^2y^5 \\
 &\mp \frac{x^8}{54000} - \frac{x^7z}{5400} \pm \frac{x^6yw}{120} + \frac{x^6w^2}{72} + \frac{x^5yzw}{2} + \frac{x^5z^3}{8} + \frac{5}{6}x^4y^3w + \frac{15}{8}x^4y^2z^2 + \frac{5}{2}x^3y^4z + \frac{x^2y^6}{2} \\
 &+ O_{x,y,z,w}(9),
 \end{aligned}$$

with 4-dimensional affine symmetry algebra generated by:

$$e_1 := (1 - y \pm \frac{1}{5}u) \partial_x + (\mp \frac{1}{5}x - z) \partial_y + (-w - \frac{4}{75}u) \partial_z + (\frac{8}{75}x \pm \frac{2}{5}z) \partial_w + x \partial_u,$$

$$e_2 := -x \partial_w + (1 - y) \partial_y - z \partial_z - w \partial_w - u \partial_u,$$

$$e_3 := \frac{2}{3}u \partial_x - x \partial_y + (1 - y \mp \frac{1}{15}u) \partial_z + (\pm \frac{2}{15}x - \frac{2}{3}z) \partial_w$$

$$e_4 := \pm \frac{5}{4}x \partial_x + \frac{1}{2}u \partial_y + (-x + \frac{5}{4}z) \partial_z + (1 - y \mp \frac{5}{2}w \mp \frac{1}{15}u) \partial_w \pm \frac{5}{2}u \partial_u,$$

sharing the Lie brackets:

$$\begin{aligned} [e_1, e_2] &= 0, & [e_1, e_3] &= \mp \frac{4}{15} e_4, & [e_1, e_4] &= \pm \frac{5}{4} e_1, \\ & & [e_2, e_3] &= 0, & [e_2, e_4] &= 0, \\ & & & & [e_3, e_4] &= \mp \frac{5}{4} e_3. \quad \square \end{aligned}$$