GRIEG LECTURE 2

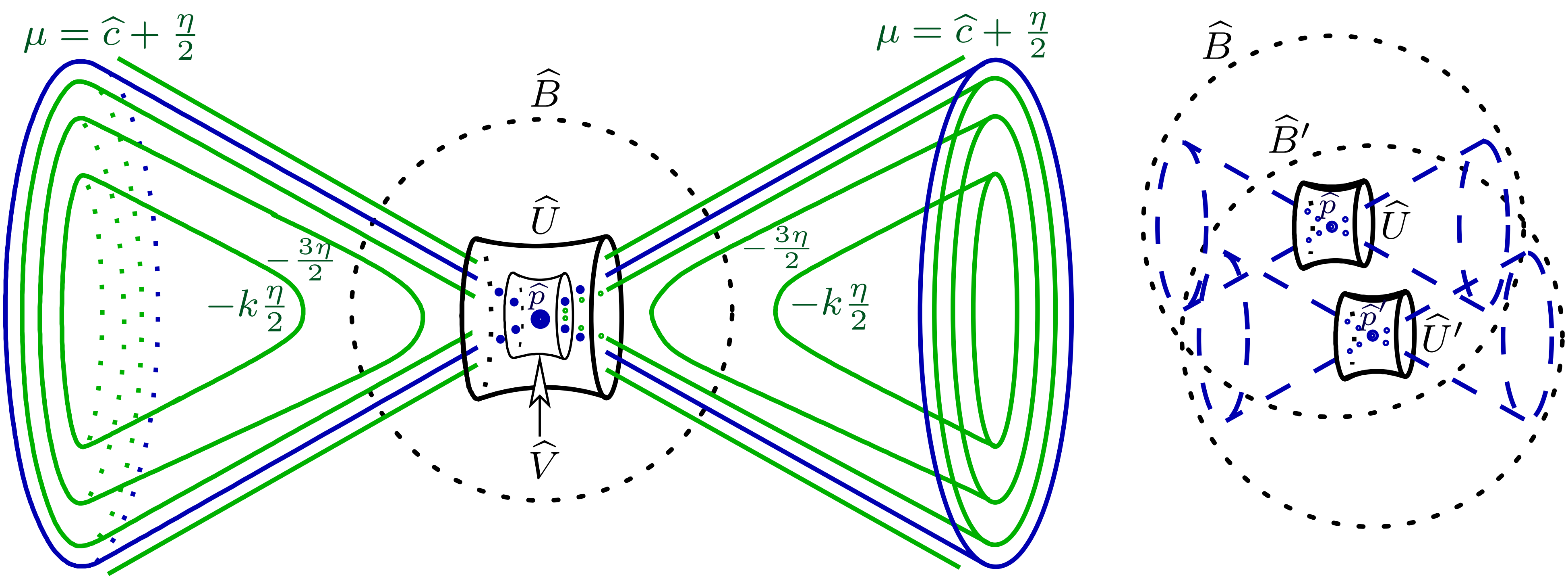
Joël Merker

www.imo.universite-paris-saclay.fr/∼merker/

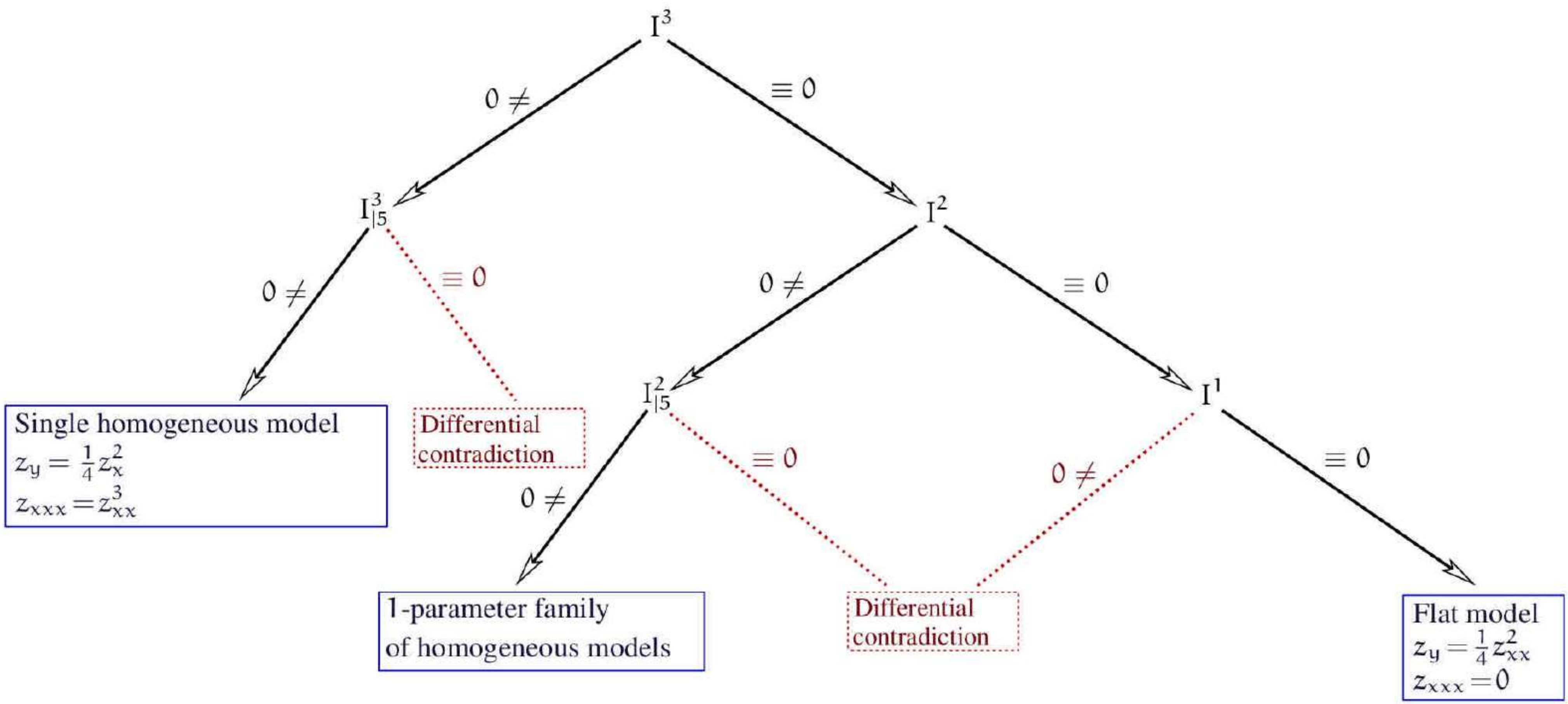
Département de Mathématique d'Orsay

Université Paris-Saclay

Friday 13 May 2022



Example of a branching tree



Second Order PDEs

- **Second order PDEs in 5D :**

$$
\begin{aligned}
z_{xx} &= F\big(x, y, z, z_x, z_y\big), \\
z_{xy} &= G\big(x, y, z, z_x, z_y\big), \\
z_{yy} &= H\big(x, y, z, z_x, z_y\big).
\end{aligned}
$$

- **Abbreviate :**

$p := z_x$ and $q := z_y$.

- **Two total differentiation operators :**

$\mathsf{D}_x := \partial_x + p\,\partial_z + F\,\partial_p + G\,\partial_q$, $\mathsf{D}_y := \partial_y + q\,\partial_z + G\,\partial_p + H\,\partial_q$.

- **Complete integrability :**

$$0 = \big[\mathsf{D}_x, \mathsf{D}_y\big],$$

that is :

$\mathsf{D}_x G = \mathsf{D}_y F$ and $\mathsf{D}_x H = \mathsf{D}_y G$.

- **First jet space :**

$$J^1 \ni (x, y, z, p, q).$$

- **Horizontal and vertical 2D distributions :**

$$\mathscr{H} := \mathsf{Span}\,\{\partial_p,\, \partial_q\}, \qquad\qquad \mathscr{V} := \mathsf{Span}\,\{\mathsf{D}_x,\, \mathsf{D}_y\}.$$

- **Nondegeneracy :**

$$\mathscr{C} := \mathscr{H} \oplus \mathscr{V}$$

of rank $2+2$ should be *contact* in J^1.

- **Terminologies :**

☐ *Para-CR structure of type* $(2, 2, 1)$ [Hill-Nurowski]

☐ *Integrable Legendrian contact structure* [Doubrov-Medvedev-The]

☐ *Submanifold of solutions* [Kamran, Makhmali, M.]

$$y = Q(x, y, a, b, c).$$

- **Flat model :**

$$z_{xx} = 0, \qquad z_{xy} = 0, \qquad z_{yy} = 0.$$

- **Symmetry projective group :**

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} := \frac{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}}{\gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 w + \delta},$$

$$\mathsf{SL}(4, \mathbb{C}) \ni \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \delta \end{pmatrix}.$$

- **Possible Lie symmetries dimensions :**

$$\mathbf{15} \qquad \mathbf{8} \qquad \mathbf{7} \qquad \mathbf{6} \qquad \mathbf{5}$$

- **Gap phenomenon :** [Kruglikov-The]

$$\mathbf{15} \downarrow \mathbf{8}.$$

- **Assertion :** *The gap phenomenon also holds for non-parabolic geometries.*

- **Submaximal model :** [Winkelmann]

$$z_{xx} = z_y^2, \qquad z_{xy} = 0, \qquad z_{yy} = 0,$$

with symmetries :

$$\begin{aligned}
&\partial_x, \qquad \partial_y, \qquad \partial_u, \qquad x\,\partial_y, \qquad x\,\partial_u, \\
&x\,\partial_x - 2u\,\partial_u, \qquad y\,\partial_y + 2u\,\partial_u, \\
&x^2\,\partial_y - y\,\partial_u.
\end{aligned}$$

- **Invariant differential quartic :** $[\mathsf{r} : \mathsf{s}] \in \mathbb{P}^1(\mathbb{C})$

$$F_{qq}\,\mathsf{r}^4 + 2\left(F_{pq} - G_{qq}\right)\mathsf{r}^3\mathsf{s} + \left(F_{pp} - 4\,G_{pq} + H_{qq}\right)\mathsf{r}^2\mathsf{s}^2 + 2\left(H_{pq} - G_{pp}\right)\mathsf{r}\mathsf{s}^3 + H_{pp}\,\mathsf{s}^4.$$

- **Characterization of flatness :**

$$\begin{aligned}
0 &\equiv F_{qq} \\
&\equiv F_{pq} - G_{qq} \\
&\equiv F_{pp} - 4\,G_{pq} + H_{qq} \\
&\equiv H_{pq} - G_{pp} \\
&\equiv H_{pp}.
\end{aligned}$$

- **Types :**

Type O : Quartic identically zero.

Type N : A single root of multiplicity 4.

Type D : Two distinct roots, each of multiplicity 2.

Type III : One root of multiplicity 3, another distinct root of multiplicity 1.

Type II : Three distinct roots, of respective multiplicities 2, 1, 1.

Type I : Four mutually distinct roots.

- **Classification of multiply transitive PDEs :**

  - ☐ [Doubrov-Medvedev-The]

Item	Model	Parameters	Symmetries
			Root type N
N.8	$u_{xx}=u_y^2$ $u_{xy}=0$ $u_{yy}=0$		$\partial_x,\ \partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x-2u\partial_u,\ y\partial_y+2u\partial_u,$ $x^2\partial_y-y\partial_u$
N.7-1a	$u_{xx}=x^\kappa u_y^2$ $u_{xy}=0$ $u_{yy}=0$	$\kappa\neq-1,-2,0,-3$	$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $y\partial_y+2u\partial_u,\ x\partial_x+\kappa y\partial_y+(\kappa-2)u\partial_u,$ $\frac{x^{\kappa+2}}{\kappa+2}\partial_y-\frac{\kappa+1}{2}y\partial_u$
N.7-1b	$u_{xx}=x^{-1}u_y^2$ $u_{xy}=0$ $u_{yy}=0$		$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $y\partial_y+2u\partial_u,\ x\partial_x-y\partial_y-3u\partial_u,$ $2x\log(x)\partial_y-y\partial_u$
N.7-1c	$u_{xx}=e^x u_y^2$ $u_{xy}=0$ $u_{yy}=0$		$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $\partial_x+y\partial_y+u\partial_u,\ y\partial_y+2u\partial_u,$ $2e^x\partial_y-y\partial_u$
N.7-2	$u_{xx}=\frac{1}{u_y}$ $u_{xy}=1$ $u_{yy}=0$		$\partial_y,\ \partial_u,\ \partial_x-\partial_u,\ \partial_y+2x\partial_u,$ $2x\partial_x-\partial_y+2u\partial_u,\ x\partial_y+x^2\partial_u,$ $x^2\partial_x+u\partial_y+x(x+2u)\partial_u$
N.6-1a	$u_{xx}=u_y^\mu$ $u_{xy}=1$ $u_{yy}=0$	$\mu\neq-1,2,0,1$	$\partial_x,\ \partial_y,\ \partial_u,$ $\partial_y+2x\partial_u,\ x\partial_y+x^2\partial_u,$ $x\partial_x+(\mu+1)y\partial_y+(\mu+2)u\partial_u$
N.6-1b	$u_{xx}=\log u_y$ $u_{xy}=1$ $u_{yy}=0$		$\partial_x,\ \partial_y,\ \partial_u,$ $\partial_y+2x\partial_u,\ x\partial_y+x^2\partial_u,$ $x\partial_x-(\frac{x}{2}-y)\partial_y+2u\partial_u$
N.6-1c	$u_{xx}=u_y\log u_y$ $u_{xy}=1$ $u_{yy}=0$		$\partial_x,\ \partial_y,\ \partial_u,$ $\partial_y+2x\partial_u,\ x\partial_y+x^2\partial_u,$ $x\partial_x-(\frac{x^2}{2}-2y)\partial_y+(3u-\frac{x^3}{3})\partial_u$
N.6-2a	$u_{xx}=x^\kappa u_y^\mu$ $u_{xy}=0$ $u_{yy}=0$	$\mu\neq-1,2,0,1$ $\kappa\neq0,-3$	$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x+(\kappa+2)y\partial_y+(\kappa+2)u\partial_u,$ $(\mu-1)y\partial_y+\mu u\partial_u$
N.6-2b	$u_{xx}=x^\kappa e^{u_y}$ $u_{xy}=0$ $u_{yy}=0$	$\kappa\neq0,-3$	$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x+(\kappa+2)y\partial_y+(\kappa+2)u\partial_u,$ $y\partial_y+(y+u)\partial_u$
N.6-2c	$u_{xx}=e^x e^{u_y}$ $u_{xy}=0$ $u_{yy}=0$		$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $\partial_x+y\partial_y+u\partial_u,$ $y\partial_y+(y+u)\partial_u$
N.6-2d	$u_{xx}=x^\kappa\log x$ $u_{xy}=0$ $u_{yy}=0$	$\kappa\neq-1,-2,0,-3$	$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x+(\kappa+2)y\partial_y+(\kappa+2)u\partial_u,$ $y\partial_y-\frac{x^{\kappa+2}}{(\kappa+1)(\kappa+2)}\partial_u$
N.6-2e	$u_{xx}=x^{-2}\log u_y$ $u_{xy}=0$ $u_{yy}=0$		$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x,$ $y\partial_y+\log x\,\partial_u$

Item	Model	Parameters	Symmetries
D.7a	$u_{xx}=u_x^2$ $u_{xy}=0$ $u_{yy}=\lambda u_y^2$	$\lambda\neq 0,-1$	$\partial_x,\ \partial_y,\ \partial_u,$ $2x\partial_x-\partial_u,\ 2y\partial_y-\frac{1}{\lambda}\partial_u,$ $x^2\partial_x-x\partial_u,\ y^2\partial_y-\frac{1}{\lambda}y\partial_u$
D.7b	$u_{xx}=u_x^2$ $u_{xy}=0$ $u_{yy}=0$		$\partial_x,\ \partial_y,\ \partial_u,$ $y\partial_y,\ y\partial_u,$ $2x\partial_x-\partial_u,\ x^2\partial_x-x\partial_u$
D.6-1	$u_{xx}=u_x^2-\frac{1}{4}u_y^4$ $u_{xy}=u_y(u_x-\frac{1}{2}u_y^2)$ $u_{yy}=u_x-\frac{1}{2}u_y^2$		$\partial_x,\ \partial_y,\ \partial_u,$ $x\partial_y-y\partial_u,\ 2x\partial_x+y\partial_y-\partial_u,$ $x^2\partial_x+xy\partial_y-(x+\frac{1}{2}y^2)\partial_u$
D.6-2a	$u_{xx}=u_x^\mu$ $u_{xy}=0$ $u_{yy}=0$	$\mu\neq 0,1,2$	$\partial_x,\ \partial_y,\ \partial_u,$ $y\partial_u,\ y\partial_y,$ $\frac{\mu-1}{\mu-2}x\partial_x+y\partial_y+u\partial_u$
D.6-2b	$u_{xx}=e^{u_x}$ $u_{xy}=0$ $u_{yy}=0$		$\partial_x,\ \partial_y,\ \partial_u,$ $y\partial_u,\ y\partial_y,$ $x\partial_x+y\partial_y+(u-x)\partial_u$
D.6-3a	$u_{xx}=\lambda u_x^2\frac{(u-u_xu_y)^{1/2}}{u^{3/2}}$ $u_{xy}=1+\lambda(u_xu_y-2u)\frac{(u-u_xu_y)^{1/2}}{u^{3/2}}$ $u_{yy}=\lambda u_y^2\frac{(u-u_xu_y)^{1/2}}{u^{3/2}}$	$\lambda\neq 0,\pm\frac{1}{2}$	$\partial_x,\ \partial_y,\ x\partial_x+u\partial_u,\ y\partial_y+u\partial_u,$ $x\partial_x+y^2\partial_y+2yu\partial_u,$ $x^2\partial_x+u\partial_y+2xu\partial_u$
D.6-3b	$u_{xx}=u_x^2(1-2u_xu_y)^{1/2}$ $u_{xy}=(u_xu_y-1)(1-2u_xu_y)^{1/2}$ $u_{yy}=u_y^2(1-2u_xu_y)^{1/2}$		$\partial_x,\ \partial_y,\ \partial_u,$ $x\partial_x-y\partial_y,$ $u\partial_y+x\partial_u,\ u\partial_x+y\partial_u$
D.6-4	$u_{xx}=0$ $u_{xy}=\frac{1+u_xu_y}{u}$ $u_{yy}=0$		$\partial_x,\ \partial_y,$ $2x\partial_x+u\partial_u,\ 2y\partial_y+u\partial_u,$ $x^2\partial_x+xu\partial_u,\ y^2\partial_y+yu\partial_y$

Root type D

Item	Model	Parameters	Symmetries	Root type III
III.6-1	$u_{xx} = \frac{u_x}{x-u_y}$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_y,\ \partial_u,$ $\partial_x + y\partial_u,\ x\partial_y + \frac{x^2}{2}\partial_u,$ $y\partial_y + u\partial_u,\ x\partial_x + u\partial_u$	
III.6-2	$u_{xx} = 2u_y(2u_x - uu_y)$ $u_{xy} = u_y^2$ $u_{yy} = 0$		$\partial_x,\ \partial_y,\ x\partial_y - \partial_u,$ $y\partial_y + u\partial_u,\ 2x\partial_x + y\partial_y - u\partial_u,$ $x^2\partial_x + xy\partial_y - (y + xu)\partial_u$	

- **Question :** *Beyond the possible root types at order 4 :*

O, N, D, III, II, I,

what is the branching diagram at orders 5, 6, ... **?**

Degenerate para-CR structures and their homogeneous models

[M.-Nuroswski 2020]

Given a $\mathscr{C}^\omega$ real hypersurface $M^5 \subset \mathbb{C}^3$ of complex-graphed equation $w = \Theta(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w})$ obtained by solving for w a real implicit equation $\rho(z_1, z_2, w, \overline{z}_1, \overline{z}_2, \overline{w}) = 0$, one can forget about complex conjugation, work over the fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and consider instead, in coordinates (x, y, z, a, b, c) a so-called *submanifold of solutions* $\mathscr{M} \subset \mathbb{K}^{2+1}_{x,y,z} \times \mathbb{K}^{2+1}_{a,b,c}$ having implicit equation $\rho(x, y, z, a, b, c) = 0$, with symmetrically $d_{x,y,z}\rho \neq 0 \neq d_{a,b,c}\rho$.

One may therefore assume $\rho_z \neq 0 \neq \rho_c$, solve for z and for c by means of the implicit function theorem, and get two equivalent *graphed* equations :

$$z = Q(x, y, a, b, c) \qquad \& \qquad c = P(a, b, x, y, z).$$

In view of the intimate relationship with PDE systems, one may think that (x, y, z) are the *variables*, while (a, b, c) are the *parameters*. Two functional relations must be identically satisfied :

$$z \equiv Q\big(x, y, a, b,\ P(a, b, x, y, z)\big) \ \ \& \ \ P\big(a, b, x, y,\ Q(x, y, a, b, c)\big) \equiv c,$$

with $Q_c \neq 0 \neq P_z$ by hypothesis, and in fact $Q_c = \frac{1}{P_z}$. Two sets of five intrinsic coordinates may hence be considered :

$$\big(x, y, a, b, c\big) \qquad \& \qquad \big(a, b, x, y, z\big).$$

The infinite group of biholomorphic transformations of $\mathbb{C}^3$ would yield, by complex conjugation, the group of anti-biholomorphic transformations, in the $(\overline{z}_1, \overline{z}_2, \overline{w})$ variables. Also, for a large class of completely integrable PDE systems, the natural infinite-dimensional group consists of *split-diffeomorphisms :*

$$\begin{aligned}(x, y, z,\, a, b, c) \longmapsto\; & \big(f(x,y,z), g(x,y,z), h(x,y,z),\, \varphi(a,b,c), \psi(a,b,c), \chi(a,b,c)\big) \\ =: \; & \big(x', y', z',\, a', b', c'\big),\end{aligned}$$

which are pairs of *uncoupled* diffeomorphisms both in the variables space and in the parameters space.

Through these transformations, both 2-dimensional foliations $\{a = a_0, b = b_0, c = c_0\}$ and $\{x = x_0, y = y_0, z = z_0\}$ are invariant. Their intersections with $\mathscr{M} = \{z = Q\} = \{c = P\}$ are spanned by *two pairs* of vector fields, firstly in coordinates (x,y,a,b,c) :

$$\begin{aligned}\mathscr{L}_a &:= \frac{\partial}{\partial a} - \frac{Q_a}{Q_c}(x,y,a,b,c)\,\frac{\partial}{\partial c}, & \mathscr{K}_x &:= \frac{\partial}{\partial x}, \\ \mathscr{L}_b &:= \frac{\partial}{\partial b} - \frac{Q_b}{Q_c}(x,y,a,b,c)\,\frac{\partial}{\partial c}, & \mathscr{K}_y &:= \frac{\partial}{\partial y},\end{aligned}$$

and secondly in coordinates (a,b,x,y,z) :

$$\begin{aligned}\mathscr{L}_a &:= \frac{\partial}{\partial a}, & \mathscr{K}_x &:= \frac{\partial}{\partial x} - \frac{P_x}{P_z}(a,b,x,y,z)\,\frac{\partial}{\partial z}, \\ \mathscr{L}_b &:= \frac{\partial}{\partial b}, & \mathscr{K}_y &:= \frac{\partial}{\partial y} - \frac{P_y}{P_z}(a,b,x,y,z)\,\frac{\partial}{\partial z}.\end{aligned}$$

However, in general, their sum :

$$\mathsf{Span}\,\big\{\mathscr{L}_a, \mathscr{L}_b\big\} \oplus \mathsf{Span}\,\big\{\mathscr{K}_x, \mathscr{K}_y\big\}$$

is *not* Frobenius-integrable, as show the four Lie brackets :

$$\begin{aligned}\big[\mathscr{K}_x, \mathscr{L}_a\big] &= \frac{-Q_cQ_{xa} + Q_aQ_{xc}}{Q_cQ_c}\,\frac{\partial}{\partial c}, & \big[\mathscr{K}_x, \mathscr{L}_b\big] &= \frac{-Q_cQ_{xb} + Q_bQ_{xc}}{Q_cQ_c}\,\frac{\partial}{\partial c}, \\ \big[\mathscr{K}_y, \mathscr{L}_a\big] &= \frac{-Q_cQ_{ya} + Q_aQ_{yc}}{Q_cQ_c}\,\frac{\partial}{\partial c}, & \big[\mathscr{K}_y, \mathscr{L}_b\big] &= \frac{-Q_cQ_{yb} + Q_bQ_{yc}}{Q_cQ_c}\,\frac{\partial}{\partial c},\end{aligned}$$

with similar formulas involving P in the other coordinates (a, b, x, y, z). This conducts to introduce *two* Levi forms, firstly with respect to parameters, having invariant matrix :

$$\text{Levi}_{\text{par}}(Q) \ := \ \begin{pmatrix} \frac{-Q_cQ_{xa}+Q_aQ_{xc}}{Q_c^2} & \frac{-Q_cQ_{xb}+Q_bQ_{xc}}{Q_c^2} \\ \frac{-Q_cQ_{ya}+Q_aQ_{yc}}{Q_c^2} & \frac{-Q_cQ_{yb}+Q_bQ_{yc}}{Q_c^2} \end{pmatrix},$$

and secondly with respect to variables, having invariant matrix :

$$\text{Levi}_{\text{var}}(P) \ := \ \begin{pmatrix} \frac{-P_zP_{ax}+P_xP_{az}}{P_z^2} & \frac{-P_zP_{ay}+P_yP_{az}}{P_z^2} \\ \frac{-P_zP_{bx}+P_xP_{bz}}{P_z^2} & \frac{-P_zP_{by}+P_yP_{bz}}{P_z^2} \end{pmatrix}.$$

• **Lemma** *One has :*

$$\text{Levi}_{\text{par}}(Q) \,=\, -\,P_y{}^{\mathtt{T}}\text{Levi}_{\text{var}}(P) \iff -\,Q_c{}^{\mathtt{T}}\text{Levi}_{\text{par}}(Q) \,=\, \text{Levi}_{\text{var}}(P).$$

• **Hint of proof** Differentiate above up to order 2, perform suitable eliminations, and obtain for $1 \leqslant i, j \leqslant 2$, with $(a_1, a_2) := (a, b)$ and

$(x_1, x_2) := (x, y)$:

$$\frac{-Q_c Q_{x_i a_j} + Q_{a_j} Q_{x_i c}}{Q_c\, Q_c} = -P_z \bigg(\frac{-P_z P_{x_i a_j} + P_{x_i} P_{a_j z}}{P_z\, P_z}\bigg).$$

As a corollary :

$$\mathrm{rank}\,\mathrm{Levi}_{\mathrm{par}}(Q) \;=\; \mathrm{rank}\,\mathrm{Levi}_{\mathrm{var}}(P).$$

So one can speak of *Levi nondegenerate*, or of *constant Levi rank* 1, submanifolds of solutions.

From the three equations :

$$z = Q, \qquad\qquad z_x = Q_x, \qquad\qquad z_y = Q_y,$$

one can solve the parameters (a, b, c) precisely when the Jacobian matrix is invertible :

$$0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} = \mathsf{det}\,\mathrm{Levi}_{\mathrm{par}}(Q).$$

But when the Levi matrix is constantly of rank 1 [our current concern], one must examine '*higher order Levi forms*', for instance by differentiating up to order 3, which conducts to :

$$
\begin{aligned}
\mathrm{Freeman}_{\mathrm{par}}(Q) \ &:= \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix}, \\
\mathrm{Freeman}_{\mathrm{var}}(P) \ &:= \begin{vmatrix} P_x & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{aax} & P_{aay} & P_{aaz} \end{vmatrix}.
\end{aligned}
$$

Indeed, under the assumption of constant Levi rank 1, and more precisely, under the following assumptions which can be met after a permutation of coordinates :

$$
\begin{vmatrix} Q_a & Q_c \\ Q_{xa} & Q_{xc} \end{vmatrix} \neq 0 \equiv \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix},
$$

it can be verified that through a split-diffeomorphism, which transforms $\{z = Q(x,y,a,b,c)\}$ into $\{z' = Q'(x',y',a',b',c')\}$, one has :

$$\frac{\begin{vmatrix} Q'_{a'} & Q'_{b'} & Q'_{c'} \\ Q'_{z'a'} & Q'_{z'b'} & Q'_{z'c'} \\ Q'_{z'z'a'} & Q'_{z'z'b'} & Q'_{z'z'c'} \end{vmatrix}}{\begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{za} & Q_{zb} & Q_{zc} \\ Q_{zza} & Q_{zzb} & Q_{zzc} \end{vmatrix}} = \frac{\begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix}^3}{\begin{vmatrix} \varphi_a & \varphi_b & \varphi_c \\ \psi_a & \psi_b & \psi_c \\ \chi_a & \chi_b & \chi_c \end{vmatrix}^1} \frac{\left(\mathscr{K}_y(g) \begin{vmatrix} Q_a & Q_c \\ Q_{za} & Q_{zc} \end{vmatrix} - \mathscr{K}_x(g) \begin{vmatrix} Q_a & Q_c \\ Q_{ya} & Q_{yc} \end{vmatrix} \right)^3}{\begin{vmatrix} \mathscr{K}_x(f) & \mathscr{K}_y(f) \\ \mathscr{K}_x(g) & \mathscr{K}_y(g) \end{vmatrix}^6 \begin{vmatrix} Q_a & Q_c \\ Q_{za} & Q_{zc} \end{vmatrix}^3},$$

and this guarantees that the nonvanishing of $\mathrm{Freeman}_{\mathrm{par}}(Q)$ is an *invariant* condition.

Of course, there is a similar formula (by symmetry) satisfied by P which shows that the nonvanishing of $\mathrm{Freeman}_{\mathrm{var}}(P)$ is also invariant. But we would like to mention that such formulas would be *untrue* without the assumption that the Levi determinant vanishes identically.

When $z = Q$ is a real hypersurface $w = \Theta(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w})$ in $\mathbb{C}^3$, with :

$$(x,y,z) := (z_1, z_2, w), \qquad\qquad (a,b,c) := (\overline{z}_1, \overline{z}_2, \overline{w}),$$

so that $Q := \Theta$ and $P := \overline{\Theta}$, it is clear that :

$$\mathrm{Freeman}_{\mathrm{var}}\big(\overline{\Theta}\big) \;=\; \overline{\mathrm{Freeman}_{\mathrm{par}}(\Theta)},$$

so that one determinant is nonzero if and only if the other is.

However, for general submanifolds of solutions, and even contrary to the 'equivalence' between the two Levi determinants expressed by a lemma above, the two Freeman determinants are *totally unrelated*. Indeed, taking for instance :

$$\begin{aligned} z \,=\, Q \;&=\; c + xa + \beta\, xxb + \gamma\, yaa + \mathrm{O}_4(x,y,a,b,c), \\ \Longleftrightarrow \qquad c \,=\, P \;&=\; z - ax - \gamma\, aay - \beta\, bxx + \mathrm{O}_4(x,y,a,b,c), \end{aligned}$$

with two *uncoupled* [free, independent] constants β, γ, we have at the origin :

$$\begin{aligned} \mathrm{Freeman}_{\mathrm{par}}(Q)\big|_0 \;&=\; \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2\beta & 0 \end{vmatrix} \;=\; 2\beta, \\ \mathrm{Freeman}_{\mathrm{var}}(P)\big|_0 \;&=\; \begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -2\gamma & 0 \end{vmatrix} \;=\; 2\gamma. \end{aligned}$$

So the much studied concept of 2-nondegeneracy for CR manifolds, when generalized to para-CR geometry, splits into *two* non-equivalent concepts.

• **Definition** A submanifold of solutions $\{z = Q(x, y, a, b, c)\} = \{c = P(a, b, x, y, z)\}$ whose Levi form is everywhere of rank 1 will be called :

• 2-*nondegenerate with respect to parameters* if $0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix} =: \Delta(Q)$;

• 2-*nondegenerate with respect to variables* if $0 \neq \begin{vmatrix} P_x & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{aax} & P_{aay} & P_{aaz} \end{vmatrix} =: \square(P)$.

Thus, if we assume constant Levi rank 1 and 2-nondegeneracy with respect to parameters :

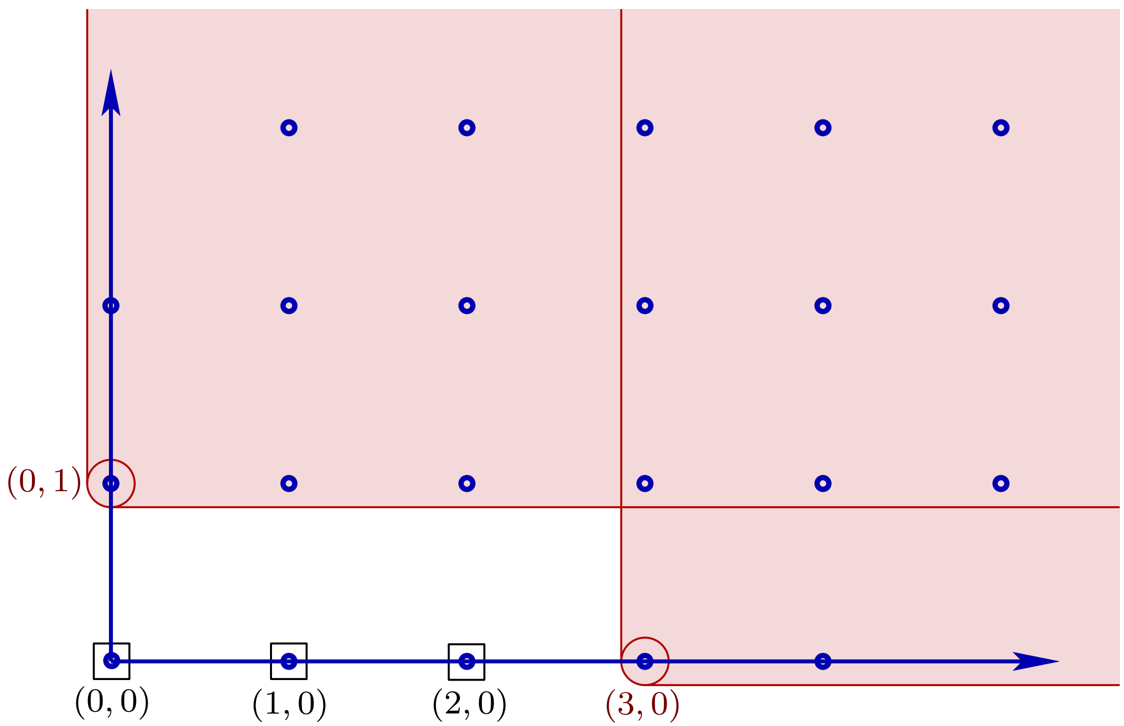
$$\begin{vmatrix} Q_a & Q_c \\ Q_{xa} & Q_{xc} \end{vmatrix} \neq 0 \equiv \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \qquad \text{and} \qquad 0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix},$$

then from the three equations $z = Q$, $z_x = Q_x$, $z_{xx} = Q_{xx}$, we can solve, by means of the implicit function theorem, the three parameters (a, b, c), namely :

$$\begin{bmatrix} z &=& Q(x,y,a,b,c), \\ z_x &=& Q_x(x,y,a,b,c), \\ z_y &=& Q_y(x,y,a,b,c), \end{bmatrix} \qquad \Longleftrightarrow \qquad \begin{bmatrix} a &=& A(x,y,z,z_x,z_{xx}), \\ b &=& B(x,y,z,z_x,z_{xx}), \\ c &=& C(x,y,z,z_x,z_{xx}), \end{bmatrix}$$

and replace in other derivatives, so that we obtain a completely integrable system of two PDEs :

$$z_y = F(x,y,z,z_x,z_{xx}) \quad \& \quad z_{xxx} = H(x,y,z,z_x,z_{xx}).$$



The transfer of derivations :

$$\begin{aligned}
\frac{\partial}{\partial z} &= A_z \frac{\partial}{\partial a} + B_z \frac{\partial}{\partial b} + C_z \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_x} &= A_{z_x} \frac{\partial}{\partial a} + B_{z_x} \frac{\partial}{\partial b} + C_{z_x} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_{xx}} &= A_{z_{xx}} \frac{\partial}{\partial a} + B_{z_{xx}} \frac{\partial}{\partial b} + C_{z_{xx}} \frac{\partial}{\partial c},
\end{aligned}$$

becomes after some elimination work :

$$\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial a} - \frac{\begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial b} + \frac{\begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_x} &= - \frac{\begin{vmatrix} Q_{b} & Q_{xxb} \\ Q_{c} & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial a} + \frac{\begin{vmatrix} Q_{a} & Q_{xxa} \\ Q_{c} & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial b} - \frac{\begin{vmatrix} Q_{a} & Q_{xxa} \\ Q_{b} & Q_{xxb} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_{xx}} &= \frac{\begin{vmatrix} Q_{b} & Q_{xb} \\ Q_{c} & Q_{xc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial a} - \frac{\begin{vmatrix} Q_{a} & Q_{xa} \\ Q_{c} & Q_{xc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial b} + \frac{\begin{vmatrix} Q_{a} & Q_{xa} \\ Q_{b} & Q_{xb} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial c}.
\end{aligned}$$

• **Lemma** *If the submanifold of solutions* $z = Q$ *has degenerate Levi form of constant rank* 1 *and if it is* 2*-nondegenerate with respect to parameters, then in its associated PDE system* $z_y = F$, $z_{xxx} = H$, *the function* F *is independent of* z_{xx} *:*

$$0 \equiv F_{z_{xx}}.$$

Proof. By construction :

$$F\big(x,y,z,z_x,z_{xx}\big) := Q_y\Big(x,y,\,A\big(x,y,z,z_x,z_{xx}\big),B\big(x,y,z,z_x,z_{xx}\big),\\ C\big(x,y,z,z_x,z_{xx}\big)\Big),$$

whence a differentiation with respect to z_{xx} make re-appear the Levi determinant :

$$\begin{aligned}
F_{z_{xx}} &= A_{z_{xx}} Q_{ya} + B_{z_{xx}} Q_{yb} + C_{z_{xx}} Q_{yc} \\
&= \frac{\begin{vmatrix} Q_b & Q_{xb} \\ Q_c & Q_{xc} \end{vmatrix}}{\Delta(Q)} Q_{ya} - \frac{\begin{vmatrix} Q_a & Q_{xa} \\ Q_c & Q_{xc} \end{vmatrix}}{\Delta(Q)} Q_{yb} + \frac{\begin{vmatrix} Q_a & Q_{xa} \\ Q_b & Q_{xb} \end{vmatrix}}{\Delta(Q)} Q_{yc} \\
&= \frac{1}{\Delta(Q)} \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \\
&\equiv 0.
\end{aligned}$$

So we do assume that F is independent of z_{xx}. After a similar work, one gets

• **Proposition** *The submanifold of solutions* $\{z = Q\}$ *is* 2*-nondegenerate with respect to* variables *if and only if :*

$$0 \neq F_{z_x z_x}. \qquad \square$$

Because it corresponds (exercise) to trivial products $\{z = Q(x, a, c)\} \times \mathbb{K}^1_y \times \mathbb{K}^1_b$, the degenerate branch $F_{z_x z_x} \equiv 0$ will not be studied, and we will constantly assume :

$$F_{z_{xx}} \equiv 0 \neq F_{z_x z_x}.$$

The graphed model :

$$z + c = \frac{2xa + x^2 b + a^2 y}{1 - yb},$$

conducts to the model PDE system :

$$z_y = \tfrac{1}{4} (z_x)^2 \quad \& \quad z_{xxx} = 0.$$

Introducing the two total differentiation operators pulled-back to the PDE system :

$$D := \partial_x + p\,\partial_z + r\,\partial_p + H\,\partial_r \quad \& \quad \Delta := \partial_y + F\,\partial_z + DF\,\partial_p + D^2 F\,\partial_r,$$

the complete integrability expresses as $D^3 F = \Delta H$, and guarantees that the general solution is of the form $Q(x, y, a, b, c)$.

Forgetting about submanifolds of solutions, we launch Cartan's method by defining a 2-nondegenerate para-CR structure on a real 5-manifold $M \ni (x, y, z, p, r)$ associated with the above two PDEs as above as an equivalence class of 1-forms modulo point equivalences in terms of an *initial* coframe of (contact) 1-forms, together with *lifted* 1-*forms*, 'rotated' by an initial G-structure :

$$
\begin{aligned}
\omega^1 &:= \mathrm{d}z - p\mathrm{d}x - F\mathrm{d}y, \\
\omega^2 &:= \mathrm{d}p - r\mathrm{d}x - DF\mathrm{d}y, \\
\omega^3 &:= \mathrm{d}r - H\mathrm{d}x - D^2F\mathrm{d}y, \\
\omega^4 &:= \mathrm{d}x, \\
\omega^5 &:= \mathrm{d}y,
\end{aligned}
\qquad
\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \end{pmatrix}
:=
\begin{pmatrix}
f^1 & 0 & 0 & 0 & 0 \\
f^2 & \rho \mathrm{e}^{\phi} & f^4 & 0 & 0 \\
f^5 & f^6 & f^7 & 0 & 0 \\
\bar{f}^2 & 0 & 0 & \rho \mathrm{e}^{-\phi} & \bar{f}^4 \\
\bar{f}^5 & 0 & 0 & \bar{f}^6 & \bar{f}^7
\end{pmatrix}
\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}.
$$

Similarly to the CR case, we perform several torsion normalizations, which lead us to change the initial coframe on M into :

$$
\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}
\longmapsto
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\frac{(2H_r^2 + 9H_p - 3DH_r)}{18} & \frac{H_r}{3} & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & F_p \\
\frac{3F_{pp}F_{pppp} - 5F_{ppp}^2}{18F_{pp}^2} & 0 & 0 & \frac{F_{ppp}}{3F_{pp}} & \frac{F_{ppp}F_p - 3F_{pp}^2}{3F_{pp}}
\end{pmatrix}
\cdot
\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix},
$$

and we invariantly reduce the G-structure to only 4 parameters ρ, ϕ, f^2, $\bar{f}^2$ (plus one extra parameter u_1), the bar having nothing to do with complex conjugation except some analogy link with Pocchiola's computations :

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \end{pmatrix} := \begin{pmatrix} \rho^2 & 0 & 0 & 0 & 0 \\ f^2 & \rho e^{\phi} & 0 & 0 & 0 \\ \frac{(f^2)^2}{2\rho^2} & \frac{f^2 e^{\phi}}{\rho} & e^{2\phi} & 0 & 0 \\ \bar{f}^2 & 0 & 0 & \rho e^{-\phi} & 0 \\ -\frac{(\bar{f}^2)^2}{2\rho^2} & 0 & 0 & \frac{-\bar{f}^2 e^{-\phi}}{\rho} & e^{-2\phi} \end{pmatrix} \cdot \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}.$$

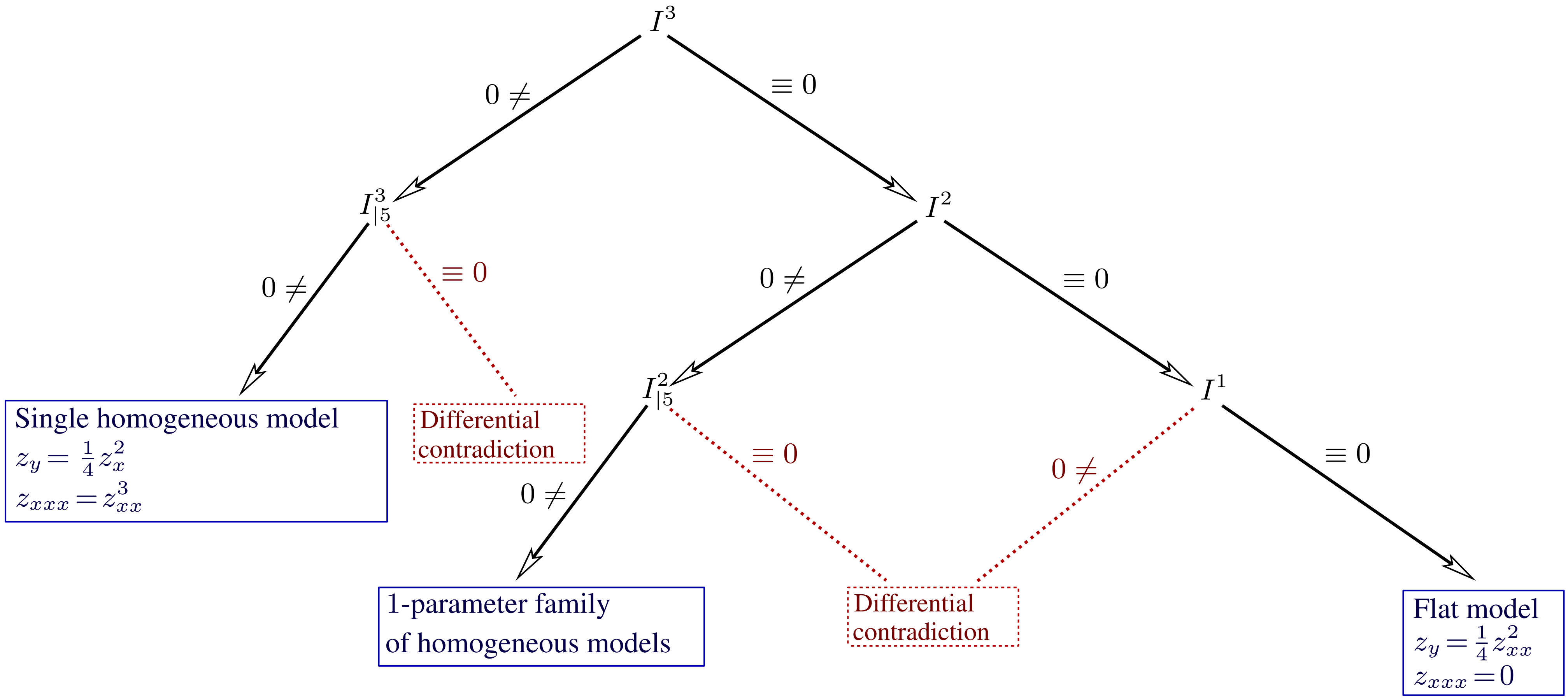
• **Theorem.** [M.-Nuroswski 2020] *On the bundle* $\mathscr{G}^9 = M^5 \times G^4$ *with* $M^5 \ni (x, y, z, p, r)$ *times* $\mathbb{R}^4 \ni (\rho, \phi, f_2, \overline{f}_2)$, *there exist four* 1*-forms* Ω_1, Ω_2, Ω_3, Ω_4 *with* θ^1, θ^2, θ^3, θ^4, θ^5, Ω_1, Ω_2, Ω_3, Ω_4 *linearly independent at* *every point which satisfy the following para-CR invariant exterior differential system :*

$$
\begin{aligned}
\mathrm{d}\theta^1 &= -\,\theta^1\wedge\Omega_1+\theta^2\wedge\theta^4,\\
\mathrm{d}\theta^2 &= \theta^2\wedge(\Omega_2-\tfrac{1}{2}\Omega_1)-\theta^1\wedge\Omega_3+\theta^3\wedge\theta^4,\\
\mathrm{d}\theta^3 &= 2\theta^3\wedge\Omega_2-\theta^2\wedge\Omega_3+\tfrac{\mathrm{e}^{3\phi}}{\rho^3}I^1\,\theta^1\wedge\theta^4+\tfrac{\mathrm{e}^{-\phi}}{\rho}I^3\,\theta^2\wedge\theta^3+\\
&\qquad \tfrac{1}{8\rho^3}\Big(2\mathrm{e}^{\phi}\bar{f}^2 {I^3}_{|5}+\rho({I^3}_{|52}+2{I^3}_{|4})-4\mathrm{e}^{-\phi}f^2I^3\Big)\,\theta^1\wedge\theta^3,\\
\mathrm{d}\theta^4 &= -\,\theta^2\wedge\theta^5-\theta^4\wedge(\tfrac{1}{2}\Omega_1+\Omega_2)-\theta^1\wedge\Omega_4,\\
\mathrm{d}\theta^5 &= -2\theta^5\wedge\Omega_2+\theta^4\wedge\Omega_4+\tfrac{\mathrm{e}^{-3\phi}}{\rho^3}I^2\,\theta^1\wedge\theta^2-\tfrac{\mathrm{e}^{\phi}}{2\rho}{I^3}_{|5}\,\theta^4\wedge\theta^5+\\
&\qquad \tfrac{1}{8\rho^3}\Big(2\mathrm{e}^{\phi}\bar{f}^2 {I^3}_{|5}+\rho({I^3}_{|52}+2{I^3}_{|4})-4\mathrm{e}^{-\phi}f^2I^3\Big)\,\theta^1\wedge\theta^5,
\end{aligned}
$$

where I^1, I^2, I^3 are explicit relative differential invariants on the base M :

$$
\begin{aligned}
I^1 &:= -\tfrac{1}{54}\big(9D^2H_r - 27DH_p - 18DH_rH_r + 18H_pH_r + 4H_r^3 + 54H_z\big),\\
I^2 &:= \frac{40F_{ppp}^3 - 45F_{pp}F_{ppp}F_{pppp} + 9F_{pp}^2F_{ppppp}}{54\,F_{pp}^3},\\
I^3 &:= \frac{2F_{ppp} + F_{pp}H_{rr}}{3\,F_{pp}},
\end{aligned}
$$

and where $(\cdot)|_i$ *for* $i = 1, \ldots, 5$ *denote directional derivatives along the vector fields* X_i *dual to* θ^i*.*



We would like to mention that when $I^3 \equiv 0$, there are striking links with the geometry of 3rd order ODEs modulo contact transformations [M.-Nurowski 2020].

Developing the technique of Cartan, we split the study in two branches : $I^3 \neq 0$ and $I^3 \equiv 0$. When $I^3 \neq 0$, we show that one can normalize ρ, u_1, $\bar{f}^2$. Then in the obtained structure equations, $I^3|_5$ becomes a relative invariant. We show that $I^3|_5 \equiv 0$ conducts to a differential contradiction. When $I^3|_5 \neq 0$, we can also normalize ϕ, f^2, hence obtaining an $\{e\}$-structure on the base M. At first, certain 15 scalar constant curvatures appear, and by looking at differential consequences of $d \circ d = 0$, they reduce to *only one pair of solutions*, with $\epsilon = \pm 1$, and we come to Maurer-Cartan type equations :

$$
\begin{aligned}
\mathrm{d}\theta^1 &= \epsilon\Big(- 6\theta^1 \wedge \theta^3 + \tfrac{1}{2}\theta^1 \wedge \theta^4 - \tfrac{3}{2}\theta^1 \wedge \theta^5\Big) + \theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^2 &= \epsilon\Big(- \tfrac{1}{16}\theta^1 \wedge \theta^2 - 2\theta^2 \wedge \theta^3 + \tfrac{1}{2}\theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^5\Big) - \theta^1 \wedge \theta^3 + \\
&\qquad \tfrac{1}{32}\theta^1 \wedge \theta^4 - \tfrac{1}{8}\theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 &= \epsilon\Big(- \tfrac{3}{16}\theta^1 \wedge \theta^3 + \tfrac{1}{2}\theta^3 \wedge \theta^4 - \tfrac{1}{2}\theta^3 \wedge \theta^5\Big) + \tfrac{1}{32}\theta^2 \wedge \theta^4 - \tfrac{1}{8}\theta^2 \wedge \theta^5, \\
\mathrm{d}\theta^4 &= \epsilon\Big(- \tfrac{1}{8}\theta^1 \wedge \theta^4 + \tfrac{1}{4}\theta^1 \wedge \theta^5 + 4\theta^3 \wedge \theta^4 - \tfrac{1}{2}\theta^4 \wedge \theta^5\Big) - \theta^2 \wedge \theta^5, \\
\mathrm{d}\theta^5 &= \epsilon\Big(- \tfrac{1}{16}\theta^1 \wedge \theta^5 + 2\theta^3 \wedge \theta^5 - \tfrac{1}{4}\theta^4 \wedge \theta^5\Big).
\end{aligned}
$$

Next, in the branch $I^3 \equiv 0$, the equations above become :

$$\begin{aligned}
\mathrm{d}\theta^1 &= -\,\theta^1 \wedge \Omega_1 + \theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^2 &= \theta^2 \wedge (\Omega_2 - \tfrac{1}{2}\Omega_1) - \theta^1 \wedge \Omega_3 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + \frac{\mathrm{e}^{3\phi}}{\rho^3} I^1 \, \theta^1 \wedge \theta^4, \\
\mathrm{d}\theta^4 &= -\,\theta^2 \wedge \theta^5 - \theta^4 \wedge (\tfrac{1}{2}\Omega_1 + \Omega_2) - \theta^1 \wedge \Omega_4, \\
\mathrm{d}\theta^5 &= -\,2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_4 + \frac{\mathrm{e}^{-3\phi}}{\rho^3} I^2 \, \theta^1 \wedge \theta^2.
\end{aligned}$$

Here, I^1 and I^2 are relative invariants.

In the sub-branch $I^2 \neq 0$, we first normalize ρ, u_1, $\bar{f}^2$. Then $I^2|_5$ becomes a relative invariant. We show that $I^2|_5 \equiv 0$ leads to a differential contradiction. When $I^2|_5 \neq 0$, we can also normalize ϕ, f^2, hence obtaining an $\{e\}$-structure on the base M. At first, certain 12 scalar constant curvatures appear, and by looking at differential consequences of $d \circ d = 0$, they reduce to *one pair of* 1*-parameter solutions* and we come to Maurer-Cartan type equations, parametrized by any $s \in \mathbb{R}$, again with $\epsilon = \pm 1$:

$$\begin{aligned}
\mathrm{d}\theta^1 &= -\,\epsilon\Big(\theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^5\Big) + \theta^2 \wedge \theta^4,\\
\mathrm{d}\theta^2 &= \epsilon\Big(s\theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5\Big) - s\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4,\\
\mathrm{d}\theta^3 &= \epsilon\Big(\theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^5\Big) - \theta^1 \wedge \theta^2 - s\theta^2 \wedge \theta^4,\\
\mathrm{d}\theta^4 &= \epsilon\Big(-\, s\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4\Big) + s\theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5,\\
\mathrm{d}\theta^5 &= \epsilon\Big(-\,\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^5\Big) + \theta^1 \wedge \theta^2 + s\theta^2 \wedge \theta^4.
\end{aligned}$$

Lastly, when $I^2 \equiv 0$, we show that $I^1 \equiv 0$ too necessarily, and we show that the structure equations are those of the model $z_y = \frac{1}{4}\,(z_{xx})^2$ & $z_{xxx} = 0$. The diagram above summarizes these explanations.

By general features of Cartan's method, all obtained para-CR structures are pairwise not equivalent.

To conclude, by setting up the PDEs associated to para-CR submanifolds of solutions inspired from Fels-Kaup's list, we realize all these homogeneous models as stated in our main

• **Theorem.** [M.-Nuroswski 2020] *Homogeneous models for 2-nondegenerate PDE five variables para-CR structures are classified by the following list of mutually inequivalent models :*

(i) $z_y = \frac{1}{4}(z_x)^2 \quad \& \quad z_{xxx} = 0\,;$

(ii) $z_y = \frac{1}{4}(z_x)^2 \quad \& \quad z_{xxx} = (z_{xx})^3\,;$

(iiia) $z_y = \frac{1}{4}(z_x)^b$ & $z_{xxx} = (2-b)\frac{(z_{xx})^2}{z_x}$ *with* $z_x > 0$ *for any real* $b \in [1,2)\,;$

(iiib) $z_y = f(z_x) \quad \& \quad z_{xxx} = h(z_x)\big(z_{xx}\big)^2$, *where the function* f *is determined by the implicit equation :*

$$\big(z_x^2 + f(z_x)^2\big)\exp\bigg(2b\arctan\frac{bz_x - f(z_x)}{z_x + bf(z_x)}\bigg) \;=\; 1 + b^2$$

and where :

$$h(z_x) \;:=\; \frac{(b^2-3)z_x - 4bf(z_x)}{\big(f(z_x) - bz_x\big)^2},$$

for any real $b > 0$.

The point automorphism groups for cases (i), (ii), (iiia), (iiib) can be determined infinitesimally. Indeed, a vector field with unknown coefficients $A^i = A^i(x, y, z, p, r)$, $i = 1, \ldots, 5$:

$$X := A^1 \, \partial_x + A^2 \, \partial_y + A^3 \, \partial_z + A^4 \, \partial_p + A^5 \, \partial_r,$$

should act on 1-forms as the matrix (2.4), so that :

$$\begin{aligned}
0 &= \mathscr{L}_X(\omega^1) \wedge \omega^1, \\
0 &= \mathscr{L}_X(\omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \\
0 &= \mathscr{L}_X(\omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \\
0 &= \mathscr{L}_X(\omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5, \\
0 &= \mathscr{L}_X(\omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5.
\end{aligned}$$

For instance, in case (ii), the first equation writes :

$$
\begin{aligned}
\mathscr{L}_x(\omega^1)\wedge\omega^1 \;=\; & \mathrm{d}x\wedge\mathrm{d}y\left[p\left(A^3_y-\tfrac{1}{4}\,p^2\,A^2_y-p\,A^1_y-\tfrac{1}{4}\,p\,A^4-\tfrac{1}{4}\,p\,A^3_x+\tfrac{1}{16}\,p^3\,A^2_x+\tfrac{1}{4}\,A^1_x\right)\right], \\
& +\mathrm{d}x\wedge\mathrm{d}z\left[p\,A^3_z-\tfrac{1}{4}\,p^3\,A^2_z-p^2\,A^1_z+A^3_x-\tfrac{1}{4}\,p^2\,A^2_x-p\,A^1_x-A^4\right] \\
& +\mathrm{d}x\wedge\mathrm{d}p\left[p\left(A^3_p-\tfrac{1}{4}\,p^2\,A^2_p-p\,A^1_p\right)\right]+\mathrm{d}x\wedge\mathrm{d}r\left[p\left(A^3_r-\tfrac{1}{4}\,p^2\,A^2_r-p\,A^1_r\right)\right] \\
& +\mathrm{d}y\wedge\mathrm{d}z\left[\tfrac{1}{4}\,p^2\,A^3_z-\tfrac{1}{16}\,p^4\,A^2_z-\tfrac{1}{4}\,p^3\,A^1_z+A^3_y-\tfrac{1}{4}\,p^2\,A^2_y-p\,A^1_y-\tfrac{1}{2}\,A^4\right] \\
& +\mathrm{d}y\wedge\mathrm{d}p\left[p^2\left(\tfrac{1}{4}\,A^3_p-\tfrac{1}{16}\,p^2\,A^2_p-\tfrac{1}{4}\,p\,A^1_p\right)\right]+\mathrm{d}y\wedge\mathrm{d}r\left[p^2\left(\tfrac{1}{4}\,A^3_r-\tfrac{1}{16}\,p^2\,A^2_r-\tfrac{1}{4}\,p\,A^1_r\right)\right] \\
& +\mathrm{d}z\wedge\mathrm{d}p\left[-A^3_p+\tfrac{1}{4}\,p^2\,A^2_p+p\,A^1_p\right]+\mathrm{d}z\wedge\mathrm{d}r\left[-A^3_r+\tfrac{1}{4}\,p^2\,A^2_r+p\,A^1_r\right].
\end{aligned}
$$

Solving this linear system of partial differential equations, we get

• **Corollary.** *The Lie algebra of infinitesimal point automorphisms of the flat model* **(i)** *is simple, isomorphic to* $\mathfrak{so}_{3,2}(\mathbb{R})$, *with the* 10 *generators :*

$$
\begin{aligned}
X_1 &:= xy\,\partial_x + y^2\,\partial_y - x^2\,\partial_z - (py + 2\,x)\,\partial_p - (2\,ry + 2)\,\partial_r, \\
X_2 &:= -\,(x^2 - yz)\,\partial_x - 2\,xy\,\partial_y - 2\,xz\,\partial_z - \big(\tfrac{1}{2}\,p^2y + 2\,z\big)\,\partial_p - \big(pry - 2\,rx + 2\,p\big)\,\partial_r, \\
X_3 &:= y\,\partial_x - 2\,x\,\partial_z - 2\,\partial_p, \\
X_4 &:= xz\,\partial_x - x^2\,\partial_y + z^2\,\partial_z - \big(\tfrac{1}{2}\,p^2x - pz\big)\,\partial_p + \big(\tfrac{1}{2}\,p^2 - prx\big)\,\partial_r, \\
X_5 &:= z\,\partial_x - 2\,x\,\partial_y - \tfrac{1}{2}\,p^2\,\partial_p - pr\,\partial_r, \\
X_6 &:= x\,\partial_x + 2\,z\,\partial_z + p\,\partial_p, \\
X_7 &:= \partial_x, \\
X_8 &:= y\,\partial_y - z\,\partial_z - p\,\partial_p - r\,\partial_r, \\
X_9 &:= \partial_y, \\
X_{10} &:= \partial_z,
\end{aligned}
$$

having commutator table :

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0	0	0	$-X_2$	0	$-X_3$	$-X_1$	$-X_6 - 2X_8$	0
X_2	$*$	0	$2X_1$	0	$2X_4$	$-X_2$	$2X_6 + 2X_8$	0	$-X_5$	$-X_3$
X_3	$*$	$*$	0	X_2	$-2X_8$	X_3	$2X_{10}$	$-X_3$	$-X_7$	0
X_4	$*$	$*$	$*$	0	0	$-2X_4$	$-X_5$	X_4	0	$-X_6$
X_5	$*$	$*$	$*$	$*$	0	$-X_5$	$2X_9$	X_5	0	$-X_7$
X_6	$*$	$*$	$*$	$*$	$*$	0	$-X_7$	0	0	$-2X_{10}$
X_7	$*$	$*$	$*$	$*$	$*$	$*$	0	0	0	0
X_8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-X_9$	X_{10}
X_9	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0
X_{10}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

In the CR context, observe that if $S^2 \subset \mathbb{R}^3$ is an affinely homogeneous parabolic surface, then the tube $M^5 := S^2 \times i\mathbb{R}^3$ has transitive holomorphic symmetry algebra $\mathfrak{hol}(M)$, with an *Abelian ideal* $\mathfrak{a} := \mathrm{Span}\,\{i\partial_{z_1}, i\partial_{z_2}, i\partial_w\}$. Conversely, for an $M^5 \in \mathfrak{C}_{2,1}$, it is not difficult to show that if $\mathfrak{hol}(M) \supset \mathfrak{a}$ contains an Abelian ideal $\mathfrak{a}$ with $\mathrm{rank}_{\mathbb{C}}\, \mathfrak{a} = 3$, then $M^5 \cong S^2 \times i\mathbb{R}^3$ is biholomorphically equivalent to the tube over an affinely homogeneous parabolic surface $S^2 \subset \mathbb{R}^3$.

In the para-CR context, all the Lie algebras in cases **(i)**, **(ii)**, **(iiia)**, **(iiib)** have a 3-dimensional abelian ideal.

• **Corollary.** *The Lie algebras of infinitesimal point automorphisms of the homogeneous models* **(ii)**, **(iiia)**, **(iiib)** *are all* 5*-dimensional and solvable, and are given in the* (x, y, z, p, r)*-space by the following generators together with their Lie brackets :*

(ii)

$$
\begin{aligned}
X_1 &:= x\,\partial_x + \tfrac{1}{2}\,y\,\partial_y + \tfrac{3}{2}\,z\,\partial_z + \tfrac{1}{2}\,p\,\partial_p - \tfrac{1}{2}\,r\,\partial_r,\\
X_2 &:= y\,\partial_x - 2\,x\,\partial_z - 2\,\partial_p,\\
X_3 &:= \partial_x,\\
X_4 &:= \partial_y,\\
X_5 &:= \partial_z,
\end{aligned}
$$

	X_1	X_2	X_3	X_4	X_5
X_1	0	$-\frac{1}{2}X_2$	$-X_3$	$-\frac{1}{2}X_4$	$-\frac{3}{2}X_5$
X_2	$*$	0	$2X_5$	$-X_3$	0
X_3	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

(iiia)

$$
\begin{aligned}
X_1 &:= x\,\partial_x + \frac{b\,z}{b-1}\,\partial_z + \frac{p}{b-1}\,\partial_p - \frac{r(b-2)}{b-1}\,\partial_r,\\
X_2 &:= y\,\partial_y - \frac{z}{b-1}\,\partial_z - \frac{p}{b-1}\,\partial_p - \frac{r}{b-1}\,\partial_z,\\
X_3 &:= \partial_x,\\
X_4 &:= \partial_y,\\
X_5 &:= \partial_z,
\end{aligned}
$$

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$-X_3$	0	$-\frac{b}{b-1}X_5$
X_2	$*$	0	0	$-X_4$	$\frac{1}{b-1}X_5$
X_3	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

(iiib)
$$\begin{aligned} X_1 &:= x\,\partial_x + y\,\partial_y + z\,\partial_z - r\,\partial_r, \\ X_2 &:= -y\,\partial_x + x\,\partial_y + \omega\,z\,\partial_z + \big(-F + \omega\,p\big)\,\partial_p + \big(-2DF + \omega\,r\big)\,\partial_r \\ X_3 &:= \partial_x, \\ X_4 &:= \partial_y, \\ X_5 &:= \partial_z, \end{aligned}$$

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$-X_3$	$-X_4$	$-X_5$
X_2	$*$	0	$-X_4$	X_3	$-\omega X_5$
X_3	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

• **Summary.**

☐ In [M.-Nurowski 2020], homogeneous models are determined by means of Cartan's equivalence method, branch by branch.

☐ Due to computational complexity, the full tree is unknown for CR manifolds of dimension 5, and/or for 5D second order nondegenerate PDEs.

• **Problem.** *For any equivalence problem, determine/construct the complete invariant branching diagram.*

• **Alternative approach :** *Power series method of equivalence.*

5D Degenerate CR Manifolds

• **5D CR hypersurface :**

$$M^5 \subset \mathbb{C}^3,$$

graphed as :

$$u = F\big(z, \zeta, \overline{z}, \overline{\zeta}\big),$$

in coordinates :

$$\big(z,\, \zeta,\, w = u + \sqrt{-1}\, v\big) \in \mathbb{C}^5.$$

• **Class of CR manifolds :**

$$\mathfrak{C}_{2,1},$$

meaning 2-nondegenerate, with Levi form of constant rank 1 < 2.

The noncylindrical parabolic surfaces $S^2 \subset \mathbb{R}^3$ can be presented following [Fels-Kaup, Acta Math. 2008] :

(1) $\big\{x_1^2 + x_2^2 = x_3^2,\ x_3 > 0\big\}$;

(2a) $\big\{r(\cos t, \sin t, e^{\omega t}) \in \mathbb{R}^3 \colon\, r \in \mathbb{R}^+$ and $t \in \mathbb{R}\big\}$ with $\omega > 0$ arbitrary ;

(2b) $\big\{r(1, t, e^t) \in \mathbb{R}^3 \colon\, r \in \mathbb{R}^+$ and $t \in \mathbb{R}\big\}$;

(2c) $\big\{r(1, e^t, e^{\theta t}) \in \mathbb{R}^3 \colon\, r \in \mathbb{R}^+$ and $t \in \mathbb{R}\big\}$ with $\theta > 2$ arbitrary ;

(3) $\big\{c(t) + r c'(t) \in \mathbb{R}^3 \colon\, r \in \mathbb{R}^+$ and $t \in \mathbb{R}\big\}$, where $c(t) = (t, t^2, t^3)$.

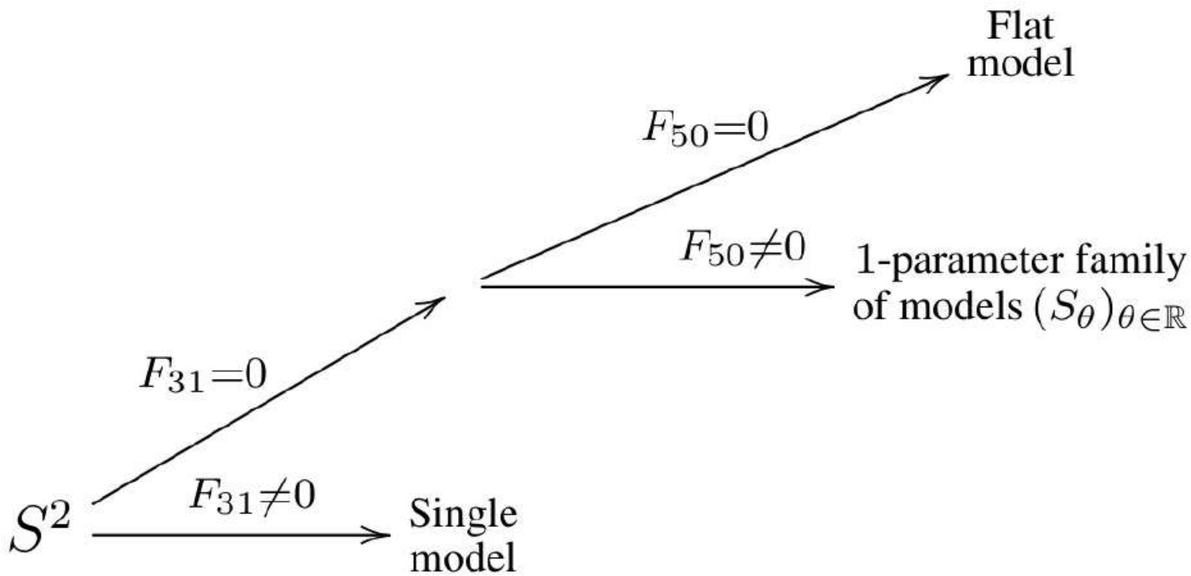
Fels-Kaup's striking result was :

• **Theorem.** *Every locally homogeneous $\mathfrak{C}_{2,1}$ hypersurface $M^5 \subset \mathbb{C}^3$ is locally biholomorphic to $S^2 \times i\mathbb{R}^3$, with $S^2 \subset \mathbb{R}^3$ being one of* **(1)**, **(2a)**, **(2b)**, **(2c)**, **(3)** *; distinct such $S^2 \times i\mathbb{R}^3$ are pairwise biholomorphically inequivalent ; all but the (flat model) tube* **(1)** *are simply transitive.*

Fels-Kaup's proof relied on expert knowledge of Lie structure theory. But only the equivalence method can reach information about CR invariants. The goal now is to explore the concerned CR invariants (either relative or absolute), since nothing about the branchings they create appears in the literature.

Affine Classification

Before stating CR results, let us present an alternative (elementary) classification of $\mathsf{Aff}(\mathbb{R}^3)$-homogeneous noncylindrical parabolic surfaces, whose final invariant tree is :



To explain this tree, let $S^2 \subset \mathbb{R}^3$ with $0 \in S^2$ be $\mathscr{C}^\omega$ graphed as $u = F(x,y) = \sum_{j+k\geqslant 1} F_{j,k} \frac{1}{j!} x^j \frac{1}{k!} y^k$. As is known, parabolicity expresses as $F_{xx} \neq 0 \equiv \left|\begin{smallmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{smallmatrix}\right|$, and non-cylindricality as $0 \neq F_{xxy} F_{xx} - F_{xxx} F_{xy}$.

A preliminary normalization is :

$$u = \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,x^2y + \tfrac{1}{6}\,F_{3,1}\,x^3y + \tfrac{1}{2}\,x^2y^2 + \mathrm{O}_{x,y}(5).$$

The coefficient $F_{3,1}$ is a relative invariant under $\mathsf{Aff}(\mathbb{R}^3)$, hence it creates a branching.

In the branch $F_{3,1} \neq 0$, one normalizes $F_{3,1} := 1$, and also $F_{4,1} := 0$, whence up to order 5 :

$$\begin{aligned} u = \ & \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,x^2y + \tfrac{1}{6}\,x^3y + \tfrac{1}{2}\,x^2y^2 \\ & + \tfrac{1}{120}\,F_{5,0}\,x^5 + \tfrac{1}{2}\,x^3y^2 + \tfrac{1}{2}\,x^2y^3 + \mathrm{O}_{x,y}(6). \end{aligned}$$

One finds that $F_{5,0} = \frac{20}{9}$ necessarily, and that all higher order $F_{j,k}$ are uniquely determined *constants*, for instance up to order 8 :

$$\begin{aligned} u = \ & \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,x^2y + \tfrac{1}{6}\,x^3y + \tfrac{1}{2}\,x^2y^2 \\ & + \tfrac{1}{54}\,x^5 + \tfrac{1}{2}\,x^3y^2 + \tfrac{1}{2}\,x^2y^3 \\ & + \tfrac{1}{162}\,x^6 + \tfrac{1}{18}\,x^5y + \tfrac{1}{8}\,x^4y^2 + x^3y^3 + \tfrac{1}{2}\,x^2y^4 \\ & - \tfrac{1}{486}\,x^7 + \tfrac{7}{108}\,x^6y + \tfrac{5}{54}\,x^5y^2 + \tfrac{5}{8}\,x^4y^3 + \tfrac{5}{3}\,x^3y^4 + \tfrac{1}{2}\,x^2y^5 \\ & + \tfrac{5}{5832}\,x^8 + \tfrac{1}{162}\,x^7y + \tfrac{1}{4}\,x^6y^2 + \tfrac{47}{216}\,x^5y^3 + \tfrac{15}{8}\,x^4y^4 + \tfrac{5}{2}\,x^3y^5 + \tfrac{1}{2}\,x^2y^6 + \mathrm{O}_{x,y}(9). \end{aligned}$$

The (transitive) affine Lie symmetry algebra is 2-dimensional, generated by :

$$\begin{aligned} e_1 &:= \big(1 + x - y - \tfrac{10}{9}\,u\big)\,\partial_x + \big(\tfrac{10}{9}\,x - y - \tfrac{10}{9}\,u\big)\,\partial_y + \big(x + 2\,u\big)\,\partial_u, \\ e_2 &:= \big(-2\,x + u\big)\,\partial_x + \big(1 - \tfrac{4}{3}\,x - y + \tfrac{8}{9}\,u\big)\,\partial_y - 3\,u\,\partial_u, \end{aligned}$$

having Lie bracket :

$$[e_1, e_2] = -\,e_1 - \tfrac{1}{3}\,e_2.$$

This is **(3)**.

Next, consider the (invariant) branch $F_{3,1} = 0$. Necessarily, $F_{4,1} = 0$, hence :

$$u = \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,x^2 y + \tfrac{1}{2}\,x^2 y^2 + \tfrac{1}{120}\,F_{5,0}\,x^5 + \tfrac{1}{2}\,x^2 y^3 + \mathrm{O}_{x,y}(6),$$

with $F_{5,0}$ being a relative invariant, again creating a (sub)branching.

The (sub)branch $F_{3,1} = 0 = F_{5,0}$ conducts to the *flat model* — a graphed representation of **(1)** above — :

$$u = \frac{1}{2}\,\frac{x^2}{1 - y},$$

having 4-dimensional (transitive) affine Lie symmetry algebra generated by :

$$\begin{aligned} e_1 &:= (1-y)\,\partial_x + x\,\partial_u, \\ e_2 &:= (1-y)\,\partial_y + u\,\partial_u, \\ e_3 &:= x\,\partial_x + 2\,u\,\partial_u, \\ e_4 &:= -\,u\,\partial_x + x\,\partial_y, \end{aligned}$$

with structure :

$$\begin{aligned} [e_1, e_2] &= e_1, \\ [e_1, e_3] &= e_1, \\ [e_1, e_4] &= e_2, \\ [e_2, e_4] &= e_4, \\ [e_3, e_4] &= e_4. \end{aligned}$$

In the thickest (sub)branch $F_{3,1} = 0 \neq F_{5,0}$, one normalizes $F_{5,0} := 1$, and also $F_{6,0} := 0$. Necessarily, $F_{5,1} = 4$ and $F_{6,1} = 0$, hence :

$$\begin{aligned} u = & \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,x^2y + \tfrac{1}{2}\,x^2y^2 + \tfrac{1}{120}\,x^5 + \tfrac{1}{2}\,x^2y^3 \\ & + \tfrac{1}{30}\,x^5y + \tfrac{1}{2}\,x^2y^4 + \tfrac{1}{5040}\,F_{7,0}\,x^7 + \tfrac{1}{12}\,x^5y^2 + \tfrac{1}{2}\,x^2y^5 + \mathrm{O}_{x,y}(8), \end{aligned}$$

with $F_{7,0}$ being an *absolute* invariant. Call it :

$$F_{7,0} \,=:\, \theta.$$

One therefore finds a 1-parameter family of affinely inequivalent homogeneous models $\big(S_\theta\big)_{\theta\in\mathbb{R}}$, with 2-dimensional (simply transitive) affine Lie symmetry algebra :

$$\begin{aligned} e_1 &:= \big(1 - y + \tfrac{1}{3}\,\theta\,u\big)\,\partial_x + \big(-\,\tfrac{1}{3}\,\theta\,x - \tfrac{1}{6}\,u\big)\,\partial_y + x\,\partial_u, \\ e_2 &:= -\,x\,\partial_x + (1-y)\,\partial_y - u\,\partial_u, \end{aligned}$$

satisfying :

$$[e_1, e_2] \,=\, 0.$$

This unifies **(2a)**, **(2b)**, **(2c)**.

CR Classification

In coordinates $\mathbb{C}^3 \ni \big(z, \zeta, w = u + \sqrt{-1}\, v\big)$, the graphed representation of the flat model is :

$$u \;=\; \frac{z\overline{z} + \frac{1}{2}\,\overline{z}^2\zeta + \frac{1}{2}\,z^2\overline{\zeta}}{1 - \zeta\overline{\zeta}} \;=:\; \boldsymbol{m}\big(z, \zeta, \overline{z}, \overline{\zeta}\big).$$

The 5-dimensional Lie group of its automorphisms fixing the origin writes :

$$z' \;:=\; \lambda\,\frac{z + i\,\alpha\,z^2 + \big(i\,\alpha\,\zeta - i\,\overline{\alpha}\big)\,w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

$$\zeta' \;:=\; \frac{\lambda}{\overline{\lambda}}\,\frac{\zeta + 2i\,\overline{\alpha}\,z - \big(\alpha\overline{\alpha} + i\,\rho\big)\,z^2 + \big(\overline{\alpha}^2 - i\,\rho\,\zeta - \alpha\overline{\alpha}\,\zeta\big)\,w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

$$w' \;:=\; \lambda\overline{\lambda}\,\frac{w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $\rho \in \mathbb{R}$ are free.

A general $\mathfrak{C}_{2,1}$ hypersurface $M^5 \subset \mathbb{C}^3$ with $0 \in M$ writes as a perturbation of this model :

$$u \;=\; F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big) \;=\; \boldsymbol{m}(z, \zeta, \overline{z}, \overline{\zeta}) + G\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big),$$

where :

$$F = \sum_{h,i,j,k,l} z^h \zeta^i \overline{z}^j \overline{\zeta}^k v^l \, F_{h,i,j,k,l} = \sum_{h,i,j,k} z^h \zeta^i \overline{z}^j \overline{\zeta}^k F_{h,i,j,k}(v),$$

with $\overline{F_{h,i,j,k,l}} = F_{j,k,h,i,l}$, with $0 = F_{0,0,0,0,0}$, and the same for G.

The Poincaré-Moser *convergent* shows that, after some local biholomorphism fixing the origin, one can assume :

$$\begin{aligned} 0 &\equiv F_{h,i,0,0}(v), & 0 &\equiv F_{3,0,0,1}(v), \\ 0 &\equiv F_{h,i,1,0}(v), & 0 &\equiv F_{4,0,0,1}(v) \equiv F_{3,0,1,1}(v), \\ 0 &\equiv F_{h,i,2,0}(v), & 0 &\equiv F_{4,0,1,1}(v) \equiv F_{3,0,3,0}(v), \end{aligned}$$

with the exceptions $1 \equiv F_{1,0,1,0}(v)$ and $\frac{1}{2} \equiv F_{2,0,0,1}(v)$.

Suppose ${M'}^5 \subset {\mathbb{C}'}^3$ is another such $\mathfrak{C}_{2,1}$ hypersurface, similarly normalized. If :

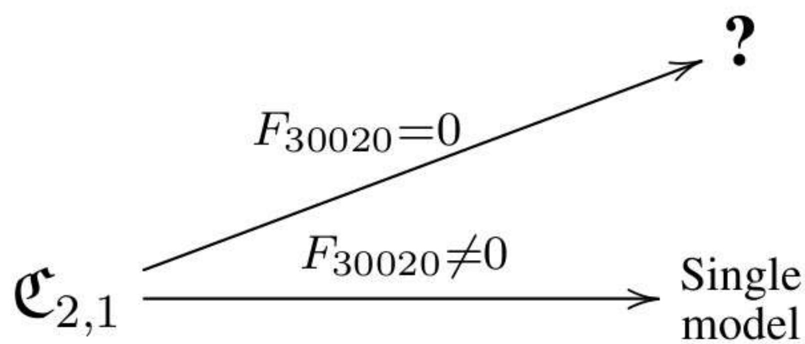
$$(z, \zeta, w) \longmapsto \big(f(z,\zeta,w),\, g(z,\zeta,w),\, h(z,\zeta,w)\big) =: (z', \zeta', w'),$$

is a local holomorphic map fixing the origin which sends M into M', then as follows from general Poincaré-Moser theory, it is of the form above for certain five real parameters $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $\rho \in \mathbb{R}$. Our goal is to normalize this remaining ambiguity.

Attributing weights $[z] := 1$, $[\zeta] := 1$, $[w] := 2$, let us therefore show weighted order 5 terms :

$$
\begin{aligned}
u = \ & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\
& + 2\,\mathsf{Re}\left\{z^3\overline{\zeta}^2\,F_{3,0,0,2,0}\right\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6),
\end{aligned}
$$

the remainder being *weighted* as well. This coefficient $F_{3,0,0,2,0}$ is a relative invariant, hence it creates a branching.



Theorem 0.1. *In the branch* $F_{3,0,0,2,0} \neq 0$*, one can normalize* $F_{3,0,0,2,0} := 1$*, so* $\lambda := 1$*, and* 3 *supplementary (real) normalizations hold :*

$$
\begin{aligned}
F_{4,0,0,2,0} &:= 0, & \textit{so}\ \ \alpha &:= 0, \\
\mathsf{Im}\,F_{3,0,2,1,0} &:= 0, & \textit{so}\ \ \rho &:= 0,
\end{aligned}
$$

so that the isotropy is reduced to be zero-dimensional.

Furthermore, all coefficients $F_{h,i,j,k,l} \in \mathbb{C}$ *are uniquely determined to be specific constants, with :*

$$F = F^2 + F^3 + F^4 + F^5 + F^6 + F^7 + F^8 + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(9),$$

where :

$$\begin{aligned}
F^2 &= z\overline{z}, \\
F^3 &= \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta}, \\
F^4 &= z\overline{z}\zeta\overline{\zeta}, \\
F^5 &= z^3\overline{\zeta}^2 + \zeta^2\overline{z}^3 + \tfrac{1}{2}\,z^2\zeta\overline{\zeta}^2 + \tfrac{1}{2}\,\zeta^2\overline{z}^2\overline{\zeta},
\end{aligned}$$

$$\begin{aligned}
F^6 = & -2\,z\zeta^2\overline{z}^3 - 2\,z^2\zeta\overline{z}^3 + 3\,z^2\zeta\overline{z}\overline{\zeta}^2 + \tfrac{1}{3}\,\zeta^3\overline{z}^3 + z\zeta^2\overline{z}\overline{\zeta}^2 - 2\,z^3\overline{z}^2\overline{\zeta} + 3\,z\zeta^2\overline{z}^2\overline{\zeta} \\
& + \tfrac{1}{3}\,z^3\,\overline{\zeta}^3 - 2\,z^3\overline{z}\overline{\zeta}^2,
\end{aligned}$$

$$\begin{aligned}
F^7 = & \ \tfrac{8}{3}\,\zeta^3\overline{z}^4 - 6\,z^2\zeta^2\overline{z}^2\overline{\zeta} + z\zeta^3\overline{z}^2\overline{\zeta} + 4\,z^3\zeta\overline{z}^3 + \tfrac{8}{3}\,z^4\overline{\zeta}^3 + z^4\overline{z}^3 + z^3\overline{z}^4 \\
& + 3\,\zeta^3\overline{z}^3\overline{\zeta} - \tfrac{4}{3}\,z^3\overline{z}\overline{\zeta}^3 - 2\,z^4\overline{z}\overline{\zeta}^2 + z^2\zeta\overline{z}\overline{\zeta}^3 - \tfrac{4}{3}\,z\zeta^3\overline{z}^3 + 2\,z^3\overline{z}^2\overline{\zeta}^2 \\
& + \tfrac{1}{2}\,\zeta^3\overline{z}^2\overline{\zeta}^2 + 3\,z^3\zeta\overline{\zeta}^3 - 6\,z^2\zeta\overline{z}^3\overline{\zeta} + \tfrac{1}{2}\,z^2\zeta^2\overline{\zeta}^3 - 6\,z^3\zeta\overline{z}^2\overline{\zeta} - 6\,z^2\zeta\overline{z}^2\overline{\zeta}^2 \\
& + 3\,z^2\zeta^2\overline{z}\overline{\zeta}^2 + 2\,z^2\zeta^2\overline{z}^3 + 3\,z\zeta^2\overline{z}^2\overline{\zeta}^2 + 4\,z^3\overline{z}^3\overline{\zeta} - 2\,z\zeta^2\overline{z}^4,
\end{aligned}$$

$$
\begin{aligned}
F^8 \,=\,\, & z\overline{z}\zeta^3\overline{\zeta}^3 + z\zeta^2\overline{z}^2\overline{\zeta}^3 + z^2\zeta^3\overline{z}\overline{\zeta}^2 + 9\,z^2\zeta^2\overline{z}\overline{\zeta}^3 + 9\,z\zeta^3\overline{z}^2\overline{\zeta}^2 + \tfrac{14}{3}\,z\zeta^3\overline{z}^3\overline{\zeta} \\
& + 12\,z^2\zeta\overline{z}^4\overline{\zeta} - 6\,z\zeta^2\overline{z}^4\overline{\zeta} - 6\,z^4\zeta\overline{z}\overline{\zeta}^2 + 9\,z^2\zeta^2\overline{z}^2\overline{\zeta}^2 + 6\,z^2\zeta\overline{z}^3\overline{\zeta}^2 + 12\,z^4\zeta\overline{z}^2\overline{\zeta} \\
& + 6\,z^3\zeta^2\overline{z}^2\overline{\zeta} + \tfrac{14}{3}\,z^3\zeta\overline{z}\overline{\zeta}^3 - 14\,z^3\zeta\overline{z}^2\overline{\zeta}^2 - 14\,z^2\zeta^2\overline{z}^3\overline{\zeta} - 4\,z^2\zeta\overline{z}^2\overline{\zeta}^3 \\
& -\,4\,z^2\zeta^3\overline{z}^2\overline{\zeta} - 6\,z^3\zeta^2\overline{z}\overline{\zeta}^2 - 6\,z\zeta^2\overline{z}^3\overline{\zeta}^2 + \zeta^3\overline{z}^3\overline{\zeta}^2 + 4\,z^3\zeta\overline{z}^4 - 12\,z^4\overline{z}\overline{\zeta}^3
\end{aligned}
$$

$$
\begin{aligned}
& + 4\,z^4\overline{z}^3\overline{\zeta} + \frac{10}{3}\,z^3\overline{z}^2\overline{\zeta}^3 + 3\,z^5\overline{z}^2\overline{\zeta} + z^3\zeta\overline{\zeta}^4 + \zeta^4\overline{z}^3\overline{\zeta} - 5\,z^4\zeta\overline{z}^3 + \frac{10}{3}\,z^2\zeta^3\overline{z}^3 \\
& -\,4\,z^4\overline{z}^2\overline{\zeta}^2 - 5\,z^3\overline{z}^4\overline{\zeta} - 12\,z\zeta^3\overline{z}^4 - 4\,z^2\zeta^2\overline{z}^4 + 3\,z^2\zeta\overline{z}^5 - \frac{2}{3}\,z^5\overline{\zeta}^3 + \frac{13}{6}\,z^4\overline{\zeta}^4 \\
& -\,2\,z^5\overline{z}^3 + \frac{13}{6}\,\zeta^4\overline{z}^4 - 2\,z^3\overline{z}^5 - \frac{2}{3}\,\zeta^3\overline{z}^5 + z^3\zeta^2\overline{\zeta}^3.
\end{aligned}
$$

The general infinitesimal CR automorphism, depending on 5 *real constants* $a, b, c, d, e \in \mathbb{R}$, *is* $L = A\,\partial_z + B\,\partial_\zeta + C\,\partial_w$, *where :*

$$
\begin{aligned}
A^0 \,&=\, a + \sqrt{-1}\,b, \\
A^1 \,&=\, \big(-a + \sqrt{-1}\,b\big)\,\zeta + \big(2\,a - 3\,c + 2\,\sqrt{-1}\,b + \sqrt{-1}\,d\big)\,z, \\
A^2 \,&=\, \big(-2\,a + 2\,c\big)\,w + \big(2\,a - 4\,c + 2\,\sqrt{-1}\,d\big)\,z^2 + \big(-c + \sqrt{-1}\,d\big)\,z\zeta, \\
A^3 \,&=\, \big(-2\,a + 2\,c\big)\,\zeta w + \big(-4\,a + 4\,c\big)\,zw,
\end{aligned}
$$

$$A^4 \;=\; 0,$$
$$A^5 \;=\; 0,$$
$$A^6 \;=\; 0,$$

where :

$$B^0 \;=\; c + \sqrt{-1}\, d,$$
$$B^1 \;=\; \left(2\sqrt{-1}\, d + 4\sqrt{-1}\, b\right) \zeta + \left(4\, a + 4\sqrt{-1}\, d\right) z,$$
$$B^2 \;=\; \left(8\, a - 8\, c + 4\sqrt{-1}\, d - 4\sqrt{-1}\, b\right) z^2 + \left(-\, 6\, a + 6\sqrt{-1}\, b - c + \sqrt{-1}\, d\right) \zeta^2 + \left(4\, a - 12\, c + 8\sqrt{-1}\, d\right) z\zeta,$$

$$B^3 \;=\; \left(4\, c - 4\sqrt{-1}\, d\right) z^3 + \left(2\sqrt{-1}\, b - 2\, a\right) \zeta^3 + \left(-\, 12\sqrt{-1}\, b + 12\, a\right) z^2\zeta + \left(-\, 8\, a + 8\, c\right) \zeta w + \left(-\, 12\sqrt{-1}\, b + 12\, a + 6\sqrt{-1}\, d - 6\, c\right) z\zeta^2,$$

$$B^4 \;=\; \left(8\, a - 8\, c\right) z^2 w + \left(-\, 6\, a + 6\sqrt{-1}\, b\right) z^4 + \left(-\, 12\sqrt{-1}\, d + 24\sqrt{-1}\, b - 24\, a + 12\, c\right) z^3\zeta + \left(8\, a - 2\, c + 2\sqrt{-1}\, d - 8\sqrt{-1}\, b\right) z\zeta^3 + \left(-\, 12\, a + 12\sqrt{-1}\, b + 12\, c - 12\sqrt{-1}\, d\right) z^2\zeta^2 + \left(-\, 12\, a + 12\, c\right) \zeta^2 w,$$

$$B^5 \;=\; \left(-\, 12\sqrt{-1}\, b + 12\, a + 6\sqrt{-1}\, d - 6\, c\right) z^5 + \left(-\, 24\, c - 30\sqrt{-1}\, b + 30\, a + 24\sqrt{-1}\, d\right) z^4\zeta + \left(-\, 20\, a + 20\sqrt{-1}\, b + 8\, c - 8\sqrt{-1}\, d\right) z^2\zeta^3 + \left(-\, 12\, c + 12\sqrt{-1}\, d\right) z^3\zeta^2 + \left(-\, 4\, a + 4\, c \right. \\ + \left(-\, 24\, c + 24\, a\right) z\zeta^2 w + \left(-\, 24\, c + 24\, a\right) z^2\zeta w,$$

and where :

$$
\begin{aligned}
C^0 &= \sqrt{-1}\, e, \\
C^1 &= \big(2\,a - 2\,\sqrt{-1}\, b\big)\, z, \\
C^2 &= \big(4\,a - 6\,c\big)\, w + \big(c - \sqrt{-1}\, d\big)\, z^2, \\
C^3 &= \big(4\,a - 4\,c\big)\, zw, \\
C^4 &= 0, \\
C^5 &= 0, \\
C^6 &= 0, \\
C^7 &= 0.
\end{aligned}
$$

and the related 5 *holomorphic vector fields* e_1, e_2, e_3, e_4, e_5 *have structure :*

$$
\begin{aligned}
[e_1, e_2] &= -\,4\,e_4 - 4\,e_5, & [e_1, e_3] &= -\,2\,e_1, & [e_1, e_4] &= 2\,e_2 + 4\,e_4, & [e_1, e_5] &= 2\,e_2 - 4\,e_5, \\
& & [e_2, e_3] &= -\,4\,e_2 - 4\,e_4, & [e_2, e_4] &= 0, & [e_2, e_5] &= 0, \\
& & & & [e_3, e_4] &= 2\,e_4, & [e_3, e_5] &= -\,2\,e_2 + 6\,e_5, \\
& & & & & & [e_4, e_5] &= 0.
\end{aligned}
$$

This Lie algebra $\mathfrak{g}$ has the derived series of dimensions 5, 4, 2, 0, with :

$$[\mathfrak{g}, \mathfrak{g}] = \mathsf{Span}\left(\underline{-4\,e_4 - 4\,e_5},\ \ -2\,e_1,\ \ \underline{2\,e_2 + 4\,e_4},\ \ \underline{-4\,e_2 - 4\,e_4}\right).$$

The three underlined vector fields span a 3-dimensional Abelian ideal $\mathfrak{a} \subset \mathfrak{g}$, whose value at the origin $0 \in \mathbb{C}^3$ spans a maximally real 3-plane. This is coherent with Fels-Kaup's item **(3)**.

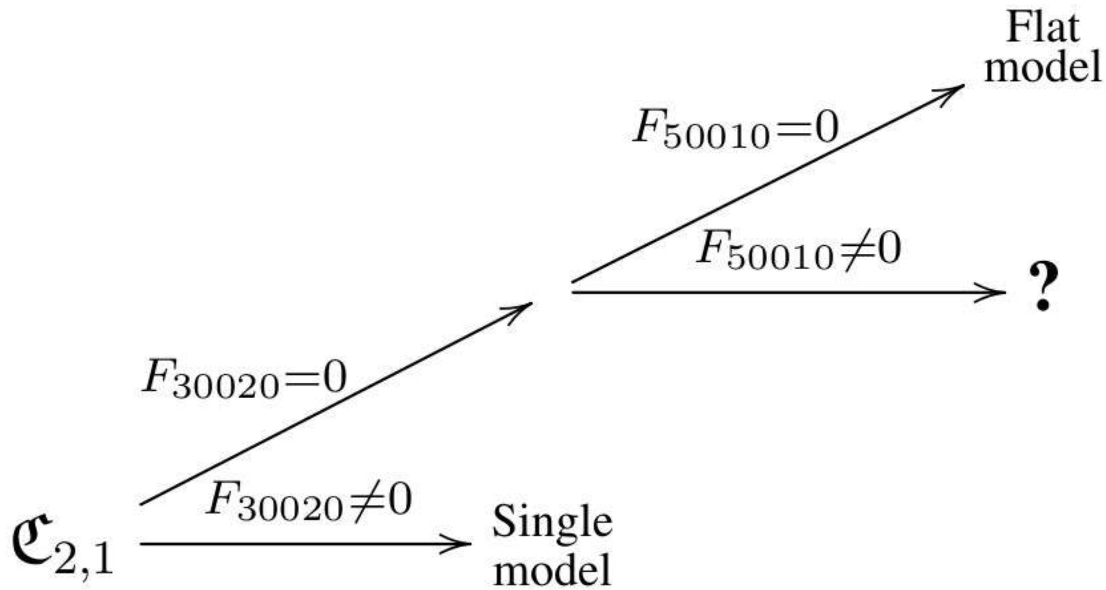
Next, assume $F_{3,0,0,2,0} \equiv 0$, or equivalently, $\frac{1}{4}\overline{\boldsymbol{W}}_0 \equiv 0$. Some differential consequences are :

$$F_{4,0,0,2,0} = 0, \qquad\qquad F_{3,0,1,2,0} = 0, \qquad\qquad F_{3,0,0,3,0} = 0,$$

hence up to order 6 :

$$\begin{aligned} u = {} & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta} \\ & + 2\,\mathsf{Re}\left\{z^5\overline{\zeta}\,F_{5,0,0,1,0} + z^3\overline{z}^2\overline{\zeta}\,F_{3,0,2,1,0}\right\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(7). \end{aligned}$$

Suppose the graphed equation for M' is similar. Then $F_{5,0,0,1,0}$ is a relative invariant, and it creates a branching :



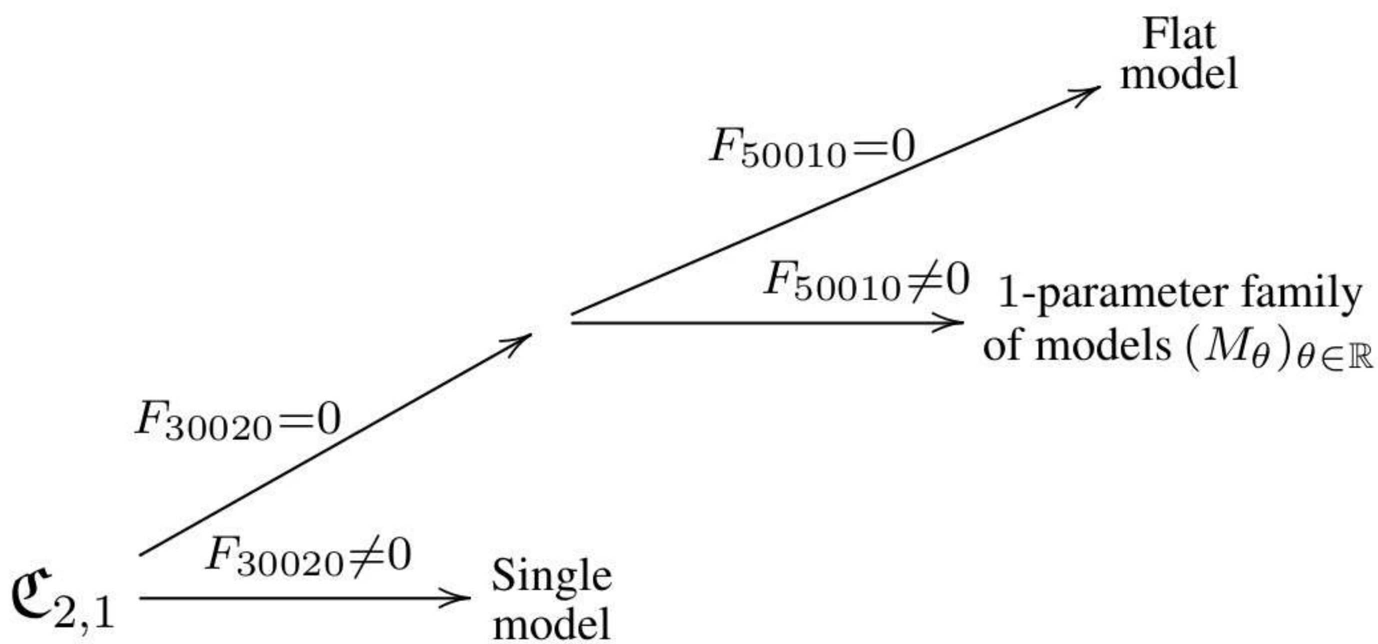
A further sub-branching could be created by the other relative invariant $F_{3,0,2,1,0}$, but this is not the case. The following result establishes, by normal forms techniques, Pocchiola's characterization of the flat model.

• **Theorem** *In the branch* $F_{3,0,0,2,0} = 0 = F_{5,0,0,1,0}$, *if* $M^5 \in \mathfrak{C}_{2,1}$ *is homogeneous, then all* $G_{h,i,j,k,l} = 0$, *and* M *coincides with the Gaussier-Merker representation of the flat model :*

$$u = \boldsymbol{m} + 0 = \frac{z\overline{z} + \frac{1}{2}\,\overline{z}^2\zeta + \frac{1}{2}\,z^2\overline{\zeta}}{1 - \zeta\overline{\zeta}}. \qquad \square$$

Thus, in this top-most (degenerate) branch, $F_{3,0,2,1,0} = 0$ is *implied*, suprisingly.

Next, in the branch $F_{3,0,0,2,0} = 0$ and $F_{5,0,0,1,0} \neq 0$, one can use $\lambda \in \mathbb{C}$ to normalize $F_{5,0,0,1,0} := 1$, so $\lambda = 1$. The final tree will be explained by the third theorem :



Theorem 0.2. *In the branch* $F_{3,0,0,2,0} = 0$ *and* $F_{5,0,0,1,0} = 1$, *three supplementary (real) normalizations hold :*

$$
\begin{aligned}
F_{6,0,0,1,0} &:= 0, && \textit{so} \ \alpha := 0, \\
\mathsf{Im}\, F_{4,0,3,0,0} &:= 0, && \textit{so} \ \rho := 0,
\end{aligned}
$$

so that the isotropy is reduced to be zero-dimensional. Notably, a constant value for $F_{3,0,2,1,0} = -15$ *is also implied.*

Furthermore, abbreviating :

$$
\theta := \mathsf{Re}\, F_{4,0,3,0,0},
$$

which is a free *absolute invariant, all coefficients* $F_{h,i,j,k,l} \in \mathbb{C}$ *are uniquely determined in terms of* $\theta \in \mathbb{R}$, *with :*

$$
F = F^2 + F^3 + F^4 + F^5 + F^6 + F^7 + F^8 + F^9 + F^{10} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(11),
$$

where :

$$
\begin{aligned}
F^2 &= z\overline{z},\\
F^3 &= \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta},\\
F^4 &= z\overline{z}\zeta\overline{\zeta},\\
F^5 &= \tfrac{1}{2}\,z^2\zeta\overline{\zeta}^2 + \tfrac{1}{2}\,\zeta^2\overline{z}^2\overline{\zeta},\\
F^6 &= -\,15\,z^3\overline{z}^2\overline{\zeta} + z\zeta^2\overline{z}\overline{\zeta}^2 + \zeta\overline{z}^5 + z^5\overline{\zeta} - 15\,z^2\zeta\overline{z}^3,
\end{aligned}
$$

$$
\begin{aligned}
F^7 = {} & \tfrac{3}{2}\,z^5\overline{\zeta}^2 + \theta\,z^4\overline{z}^3 - 45\,z^2\zeta\overline{z}^3\overline{\zeta} + \tfrac{1}{2}\,z^2\zeta^2\overline{\zeta}^3 + \theta\,z^3\overline{z}^4 - \tfrac{15}{2}\,z^4\overline{z}\overline{\zeta}^2 + 5\,z\zeta\overline{z}^4\overline{\zeta}\\
& -\;10\,z^3\overline{z}^2\overline{\zeta}^2 + \tfrac{3}{2}\,\zeta^2\overline{z}^5 - 45\,z^3\zeta\overline{z}^2\overline{\zeta} - \tfrac{15}{2}\,z\zeta^2\overline{z}^4 - 10\,z^2\zeta^2\overline{z}^3 + \tfrac{1}{2}\,\zeta^3\overline{z}^2\overline{\zeta}^2\\
& +5\,z^4\zeta\overline{z}\overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
F^8 = {} & z\zeta^3\overline{z}\overline{\zeta}^3 - 20\,z^3\zeta^2\overline{z}^2\overline{\zeta} - 75\,z^3\zeta\overline{z}^2\overline{\zeta}^2 - 75\,z^2\zeta^2\overline{z}^3\overline{\zeta} - \tfrac{75}{2}\,z\zeta^2\overline{z}^4\overline{\zeta}\\
& -\;\tfrac{75}{2}\,z^4\zeta\overline{z}\overline{\zeta}^2 - 20\,z^2\zeta\overline{z}^3\overline{\zeta}^2 - \tfrac{1}{5}\,\theta\,z^6\overline{z}\overline{\zeta} + 2\,\theta\,z^4\zeta\overline{z}^3 + \tfrac{12}{5}\,\theta\,z^5\overline{z}^2\overline{\zeta}\\
& +3\,\theta\,z^4\overline{z}^3\overline{\zeta} + 2\,\theta\,z^3\overline{z}^4\overline{\zeta} + 3\,\theta\,z^3\zeta\overline{z}^4 + \tfrac{12}{5}\,\theta\,z^2\zeta\overline{z}^5 - \tfrac{1}{5}\,\theta\,z\zeta\overline{z}^6 - 5\,z^4\overline{z}\overline{\zeta}^3\\
& -\;5\,z\zeta^3\overline{z}^4 + 5\,\zeta^2\overline{z}^5\overline{\zeta} + 5\,z^5\zeta\overline{\zeta}^2 - \tfrac{1}{35}\,\theta\,\zeta\overline{z}^7 - \tfrac{1}{35}\,\theta\,z^7\overline{\zeta} - \tfrac{3}{2}\,z^5\overline{\zeta}^3 - 130\,z^5\overline{z}^3\\
& -\;\tfrac{325}{6}\,z^4\overline{z}^4 - 130\,z^3\overline{z}^5 - \tfrac{3}{2}\,\zeta^3\overline{z}^5,
\end{aligned}
$$

$$
\begin{aligned}
F^9 \;=\;& \theta\, z^3\overline{z}^4\overline{\zeta}^2 + \theta\, z^4\zeta^2\overline{z}^3 - 165\, z^2\zeta^2\overline{z}^3\overline{\zeta}^2 - 40\, z^3\zeta\overline{z}^2\overline{\zeta}^3 - 165\, z^3\zeta^2\overline{z}^2\overline{\zeta}^2 \\
& - 5\, z^4\zeta^2\overline{z}\overline{\zeta}^2 - 5\, z\zeta^2\overline{z}^4\overline{\zeta}^2 - \tfrac{75}{2}\, z^4\zeta\overline{z}\overline{\zeta}^3 - \tfrac{75}{2}\, z\zeta^3\overline{z}^4\overline{\zeta} - 40\, z^2\zeta^3\overline{z}^3\overline{\zeta} \\
& + \tfrac{18}{5}\,\theta\, z^5\overline{z}^2\overline{\zeta}^2 + 3\,\theta\, z^4\overline{z}^3\overline{\zeta}^2 + 2\,\theta\, z^6\overline{z}\overline{\zeta}^2 + \tfrac{18}{5}\,\theta\, z^2\zeta^2\overline{\zeta}^5 + 2\,\theta\, z\zeta^2\overline{z}^6 \\
& + 3\,\theta\, z^3\zeta^2\overline{\zeta}^4 - 5\sqrt{-1}\, z^6\overline{\zeta}v + 5\sqrt{-1}\,\zeta\overline{z}^6 v + \tfrac{24}{5}\,\theta\, z^2\zeta\overline{z}^5\overline{\zeta} + 12\,\theta\, z^4\zeta\overline{z}^3\overline{\zeta} \\
& + \tfrac{24}{5}\,\theta\, z^5\zeta\overline{z}^2\overline{\zeta} + 12\,\theta\, z^3\zeta\overline{z}^4\overline{\zeta} - \tfrac{1}{5}\,\theta\, z\zeta\overline{z}^6\overline{\zeta} - \tfrac{1}{5}\,\theta\, z^6\zeta\overline{z}\overline{\zeta} - 100\sqrt{-1}\, z^3\overline{z}^3\overline{\zeta}v
\end{aligned}
$$

$$
\begin{aligned}
& - 30\sqrt{-1}\, z^5\overline{z}\overline{\zeta}v - 75\sqrt{-1}\, z^4\overline{z}^2\overline{\zeta}v - \tfrac{335}{3}\, z^5\overline{z}^3\overline{\zeta} + 5\, z\zeta\overline{z}^7 - \tfrac{6}{35}\,\theta\,\zeta^2\overline{\zeta}^7 - \tfrac{335}{3}\, z^3\zeta\overline{z}^5 \\
& + \tfrac{1}{2}\, z^2\zeta^3\overline{\zeta}^4 + \tfrac{475}{2}\, z^4\overline{z}^4\overline{\zeta} - 455\, z^2\zeta\overline{z}^6 - \tfrac{6}{25}\,\theta^2\, z^5\overline{z}^4 - 190\, z^5\zeta\overline{z}^3 - \tfrac{6}{35}\,\theta\, z^7\overline{\zeta}^2 \\
& + \tfrac{1}{2}\,\zeta^4\overline{z}^2\overline{\zeta}^3 - 455\, z^6\overline{z}^2\overline{\zeta} + 5\, z^7\overline{z}\overline{\zeta} - \tfrac{6}{25}\,\theta^2\, z^4\overline{z}^5 - 190\, z^3\overline{z}^5\overline{\zeta} - \tfrac{4}{25}\,\theta^2\, z^6\overline{z}^3 \\
& - \tfrac{4}{25}\,\theta^2\, z^3\overline{z}^6 - \tfrac{15}{2}\, z^5\zeta\overline{\zeta}^3 - \tfrac{15}{2}\,\zeta^3\overline{z}^5\overline{\zeta} - z^5\overline{\zeta}^4 - \zeta^4\overline{z}^5 + \tfrac{25}{4}\,\zeta\overline{z}^8 + \tfrac{25}{4}\, z^8\overline{\zeta} \\
& + 30\sqrt{-1}\, z\zeta\overline{z}^5 v + 100\sqrt{-1}\, z^3\zeta\overline{z}^3 v + 75\sqrt{-1}\, z^2\zeta\overline{z}^4 v + \tfrac{475}{2}\, z^4\zeta\overline{z}^4,
\end{aligned}
$$

$$
\begin{aligned}
F^{10} \;=\;& z\zeta^4\overline{z}\overline{\zeta}^4 + \theta\, z^3\zeta^3\overline{z}^4 + \theta\, z^4\overline{z}^3\overline{\zeta}^3 - 210\, z^3\zeta^2\overline{z}^2\overline{\zeta}^3 - 20\, z^4\zeta\overline{z}\overline{\zeta}^4 + 105\, z^5\zeta\overline{z}^3\overline{\zeta} \\
& - 210\, z^2\zeta^3\overline{z}^3\overline{\zeta}^2 - \tfrac{1525}{2}\, z^2\zeta\overline{z}^6\overline{\zeta} - \tfrac{1525}{2}\, z^6\zeta\overline{z}^2\overline{\zeta} - \tfrac{255}{2}\, z^4\zeta^2\overline{z}\overline{\zeta}^3 \\
& + 1725\, z^4\zeta\overline{z}^4\overline{\zeta} - 20\, z\zeta^4\overline{z}^4\overline{\zeta} - 70\, z^3\overline{\zeta}^3\overline{z}^2\overline{\zeta}^2 - \tfrac{255}{2}\, z\zeta^3\overline{z}^4\overline{\zeta}^2 - 70\, z^2\zeta^2\overline{z}^3\overline{\zeta}^3 \\
& + 105\, z^3\zeta\overline{z}^5\overline{\zeta} + 50\, z^7\zeta\overline{z}\overline{\zeta} + 50\, z\zeta\overline{z}^7\overline{\zeta} + \tfrac{3}{175}\,\theta^2\, z\zeta\overline{z}^8 + \tfrac{3}{175}\,\theta^2\, z^8\overline{z}\overline{\zeta}
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{4}{5}\,\theta^2\, z^3\zeta\overline{z}^6 + 2\,\theta\, z\zeta^3\overline{z}^6 + 2\,\theta\, z^6\overline{z}\overline{\zeta}^3 + \tfrac{12}{5}\,\theta\, z^5\overline{z}^2\overline{\zeta}^3 - \tfrac{18}{25}\,\theta^2\, z^5\zeta\overline{z}^4\\
&- \tfrac{8}{25}\,\theta^2\, z^6\zeta\overline{z}^3 - \tfrac{18}{25}\,\theta^2\, z^4\overline{z}^5\overline{\zeta} - \tfrac{8}{25}\,\theta^2\, z^3\overline{z}^6\overline{\zeta} - \tfrac{72}{175}\,\theta^2\, z^2\zeta\overline{\zeta}^7 - \tfrac{4}{5}\,\theta^2\, z^6\overline{z}^3\overline{\zeta}\\
&- \tfrac{24}{25}\,\theta^2\, z^5\overline{z}^4\overline{\zeta} - \tfrac{24}{25}\,\theta^2\, z^4\zeta\overline{z}^5 + \tfrac{12}{5}\,\theta\, z^2\zeta^3\overline{z}^5 - \tfrac{72}{175}\,\theta^2\, z^7\overline{z}^2\overline{\zeta} - \tfrac{1}{5}\,\theta\,\zeta^2\overline{z}^7\overline{\zeta}\\
&- \tfrac{1}{5}\,\theta\, z^7\zeta\overline{\zeta}^2 - 20\,\sqrt{-1}\, z^6\overline{\zeta}v + 20\,\sqrt{-1}\,\zeta^2\overline{z}^6 v + 24\,\theta\, z^3\zeta^2\overline{z}^4\overline{\zeta} + \tfrac{18}{5}\,\theta\, z^6\zeta\overline{z}\overline{\zeta}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{12}{5}\,\theta\, z^2\zeta\overline{z}^5\overline{\zeta}^2 + \tfrac{18}{5}\,\theta\, z\zeta^2\overline{z}^6\overline{\zeta} + 15\,\theta\, z^4\zeta^2\overline{z}^3\overline{\zeta} + 15\,\theta\, z^3\zeta\overline{z}^4\overline{\zeta}^2\\
&+ \tfrac{12}{5}\,\theta\, z^5\zeta^2\overline{z}^2\overline{\zeta} + 18\,\theta\, z^2\zeta^2\overline{z}^5\overline{\zeta} + 18\,\theta\, z^5\zeta\overline{z}^2\overline{\zeta}^2 + 24\,\theta\, z^4\,\zeta\overline{z}^3\overline{\zeta}^2\\
&- 150\,\sqrt{-1}\, z^4\overline{z}^2\overline{\zeta}^2 v - 16\,\sqrt{-1}\,\theta\, z^3\overline{z}^5 v - 100\,\sqrt{-1}\, z^3\overline{z}^3\overline{\zeta}^2 v - 90\,\sqrt{-1}\, z^5\overline{z}\overline{\zeta}^2 v + F_{5,0,5,0,0}\, z^5\overline{z}^5\\
&- \tfrac{15}{2}\, z^5\zeta\overline{\zeta}^4 + 5\,\zeta^3\overline{z}^5\overline{\zeta}^2 + 5\, z^5\zeta^2\overline{\zeta}^3 - 150\, z^5\zeta^2\overline{\zeta}^3 + \tfrac{975}{2}\, z^4\zeta^2\overline{z}^4 + 570\, z^3\zeta^2\overline{z}^5
\end{aligned}
$$

$$
\begin{aligned}
&- 325\, z^2\zeta^2\overline{z}^6 - 435\, z\zeta^2\overline{z}^7 - 435\, z^7\overline{z}\overline{\zeta}^2 - 325\, z^6\overline{z}^2\overline{\zeta}^2 - \tfrac{15}{2}\,\zeta^4\overline{z}^5\overline{\zeta}\\
&+ 570\, z^5\overline{z}^3\overline{\zeta}^2 + \tfrac{975}{2}\, z^4\overline{z}^4\overline{\zeta}^2 - 150\, z^3\overline{z}^5\overline{\zeta}^2 + 9\,\theta\, z^4\overline{z}^6 + \tfrac{164}{7}\,\theta\, z^3\overline{z}^7 + \tfrac{4}{7}\,\theta\,\zeta^3\overline{z}^7\\
&+ \tfrac{1}{525}\,\theta^2\,\zeta\overline{z}^9 + \tfrac{4}{7}\,\theta\, z^7\overline{\zeta}^3 + \tfrac{1}{525}\,\theta^2\, z^9\overline{\zeta} + 9\,\theta\, z^6\overline{z}^4 + \tfrac{164}{7}\,\theta\, z^7\overline{z}^3 + \tfrac{95}{4}\,\zeta^2\overline{z}^8\\
&+ \tfrac{95}{4}\, z^8\overline{\zeta}^2 + 90\,\sqrt{-1}\, z\zeta^2\overline{z}^5 v + 100\,\sqrt{-1}\, z^3\zeta^2\overline{z}^3 v + 150\,\sqrt{-1}\, z^2\zeta^2\overline{z}^4 v + 16\,\sqrt{-1}\,\theta\, z^5\overline{z}^3 v\\
&- 30\,\sqrt{-1}\, z^5\zeta\overline{z}\overline{\zeta}v - 150\,\sqrt{-1}\, z^2\zeta\overline{z}^4\overline{\zeta}v + 150\,\sqrt{-1}\, z^4\zeta\overline{z}^2\overline{\zeta}v + 30\,\sqrt{-1}\, z\zeta\overline{z}^5\overline{\zeta}v.
\end{aligned}
$$

The general infinitesimal CR automorphism, depending on 5 *real constants* $a, b, c, d, e \in \mathbb{R}$*, is* $L = A\,\partial_z + B\,\partial_\zeta + C\,\partial_w$*, where :*

$$
\begin{aligned}
A^0 &= a + \sqrt{-1}\,b,\\
A^1 &= \big(-c + \sqrt{-1}\,d\big)\,z + \big(-a + \sqrt{-1}\,b\big)\,\zeta,\\
A^2 &= \big(\tfrac{2}{5}\,\theta\,a + 5\sqrt{-1}\,e\big)\,z^2 + \big(-\tfrac{2}{5}\,\theta\,a + 5\sqrt{-1}\,e\big)\,w + \big(-c + \sqrt{-1}\,d\big)\,z\zeta,\\
A^3 &= \big(-10\,a - 10\sqrt{-1}\,b\big)\,z^3 + \big(10\sqrt{-1}\,b + 30\,a\big)\,zw + \big(-\tfrac{2}{5}\,\theta\,a - 5\sqrt{-1}\,e\big)\,\zeta w,
\end{aligned}
$$

$$
\begin{aligned}
A^4 &= \big(-10\,c - 5\sqrt{-1}\,d\big)\,w^2 + \big(10\,a + 10\sqrt{-1}\,b\big)\,z\zeta w + \big(-20\,c + 10\sqrt{-1}\,d\big)\,z^2w,\\
A^5 &= \big(-\tfrac{4}{5}\,\theta\,a - 10\sqrt{-1}\,e\big)\,z^5 + \big(-5\,c + 5\sqrt{-1}\,d\big)\,z^4\zeta + \big(4\,\theta\,a - 50\sqrt{-1}\,e\big)\,z^3w\\
&\quad + \big(75\sqrt{-1}\,e - 6\,\theta\,a\big)\,zw^2 + \big(10\,c - 5\sqrt{-1}\,d\big)\,\zeta w^2,
\end{aligned}
$$

$$
\begin{aligned}
A^6 &= \big(-20\,a - 20\sqrt{-1}\,b - \tfrac{1}{5}\sqrt{-1}\,\theta\,d + \tfrac{1}{5}\,\theta\,c\big)\,z^6 + \big(-\tfrac{200}{3}\,a + \tfrac{100}{3}\sqrt{-1}\,b\big)\,w^3\\
&\quad + \big(-2\,\theta\,a + 25\sqrt{-1}\,e\big)\,z\zeta w^2 + \big(200\,a + 100\sqrt{-1}\,b\big)\,z^2w^2,
\end{aligned}
$$

$$
\begin{aligned}
A^7 &= \big(10\sqrt{-1}\,d - 10\,c + \tfrac{2}{7}\sqrt{-1}\,\theta\,e + \tfrac{4}{175}\,\theta^2\,a\big)\,z^7 + \big(100\,c + 50\sqrt{-1}\,d\big)\,z^3w^2\\
&\quad + \big(-\tfrac{200}{3}\,a - \tfrac{100}{3}\sqrt{-1}\,b\big)\,\zeta w^3 + \big(-\tfrac{1}{5}\sqrt{-1}\,\theta\,d + \tfrac{1}{5}\,\theta\,c\big)\,z^6\zeta + \big(50\,a + 50\sqrt{-1}\,b\big)\,z^4\zeta w\\
&\quad + \big(-50\sqrt{-1}\,d + 70\,c\big)\,z^5w + \big(-100\,c - 50\sqrt{-1}\,d + 20\sqrt{-1}\,\theta\,e\big)\,zw^3,
\end{aligned}
$$

$$
\begin{aligned}
A^8 &= A_{0,0,4}\,w^4 + \big(-\tfrac{3}{175}\,\theta^2\,c + \tfrac{4}{7}\sqrt{-1}\,\theta\,b - \tfrac{31}{7}\,\theta\,a + \tfrac{3}{175}\sqrt{-1}\,\theta^2\,d - \tfrac{125}{2}\sqrt{-1}\,e\big)\,z^8\\
&\quad + \big(8\,\overline{A_{0,0,4}} + \tfrac{1}{2}\,\overline{B_{1,0,3}}\big)\,z^2w^3 + \big(-30\sqrt{-1}\,d + 30\,c\big)\,z^5\zeta w + \big(-\tfrac{100}{3}\,c - \tfrac{50}{3}\sqrt{-1}\,d\big)\,z\zeta w^3\\
&\quad + \big(-50\,c + 50\sqrt{-1}\,d\big)\,z^7\zeta + \big(-2\sqrt{-1}\,\theta\,b + 10\,\theta\,a - 50\sqrt{-1}\,e\big)\,z^6w,
\end{aligned}
$$

where :

$$
\begin{aligned}
B^0 &= c + \sqrt{-1}\, d, \\
B^1 &= \big(\tfrac{4}{5}\,\theta\, a - 10\,\sqrt{-1}\, e\big)\, z + \big(2\,\sqrt{-1}\, d\big)\, \zeta, \\
B^2 &= \big(-40\,\sqrt{-1}\, b - 60\, a\big)\, z^2 + \big(-c + \sqrt{-1}\, d\big)\, \zeta^2 + \big(10\, a - 10\,\sqrt{-1}\, b\big)\, w + \big(\tfrac{4}{5}\,\theta\, a + 10\,\sqrt{-1}\, e\big) \\
B^3 &= \big(-30\,\sqrt{-1}\, d + 30\, c\big)\, z^3 + \big(40\, c + 20\,\sqrt{-1}\, d\big)\, zw + \big(60\, a\big)\, \zeta w + \big(40\, a - 140\,\sqrt{-1}\, b\big)\, z^2\zeta,
\end{aligned}
$$

$$
\begin{aligned}
B^4 = {} & \big(-14\,\theta\, a + 100\,\sqrt{-1}\, e + 6\,\sqrt{-1}\,\theta\, b\big)\, z^4 + \big(2\,\theta\, a + 25\,\sqrt{-1}\, e\big)\, w^2 + \big(-40\, c + 20\,\sqrt{-1}\, d\big)\, z\zeta \\
& + \big(24\,\theta\, a - 300\,\sqrt{-1}\, e\big)\, z^2 w + \big(10\, a + 10\,\sqrt{-1}\, b\big)\, \zeta^2 w + \big(-90\,\sqrt{-1}\, d + 90\, c\big)\, z^3\zeta \\
& + \big(-60\,\sqrt{-1}\, b + 60\, a\big)\, z^2\zeta^2,
\end{aligned}
$$

$$
\begin{aligned}
B^5 = {} & \big(-860\,\sqrt{-1}\, b + 900\, a + \tfrac{24}{5}\,\sqrt{-1}\,\theta\, d - \tfrac{24}{5}\,\theta\, c\big)\, z^5 + \big(40\, c - 40\,\sqrt{-1}\, d\big)\, z^3\zeta^2 \\
& + \big(12\,\sqrt{-1}\,\theta\, b - 20\,\theta\, a - 100\,\sqrt{-1}\, e\big)\, z^4\zeta + \big(-300\, a - 300\,\sqrt{-1}\, b\big)\, z^3 w \\
& + \big(400\, a - 200\,\sqrt{-1}\, b\big)\, zw^2 + \big(150\,\sqrt{-1}\, e\big)\, \zeta w^2 + \big(56\,\theta\, a + 200\,\sqrt{-1}\, e\big)\, z^2\zeta w,
\end{aligned}
$$

$$
\begin{aligned}
B^6 = {} & \big(-770\,\sqrt{-1}\, d + 690\, c + 4\,\sqrt{-1}\,\theta\, e + \tfrac{32}{25}\,\theta^2\, a - \tfrac{24}{25}\,\sqrt{-1}\,\theta^2\, b\big)\, z^6 \\
& + \big(-\tfrac{100}{3}\, c + \tfrac{50}{3}\,\sqrt{-1}\, d\big)\, w^3 + \big(-250\, c - 350\,\sqrt{-1}\, d - 30\,\sqrt{-1}\,\theta\, e - \tfrac{12}{5}\,\theta^2\, a\big)\, z^4 w \\
& + \big(400\, a + 200\,\sqrt{-1}\, b\big)\, z\zeta w^2 + \big(-2\,\theta\, a + 25\,\sqrt{-1}\, e\big)\, \zeta^2 w^2 + \big(-6\,\theta\, a + 6\,\sqrt{-1}\,\theta\, b\big)\, z^4\zeta^2 \\
& + \big(600\, c + 300\,\sqrt{-1}\, d - 60\,\sqrt{-1}\,\theta\, e\big)\, z^2 w^2 + \big(-1500\, a - 300\,\sqrt{-1}\, b\big)\, z^3\zeta w \\
& + \big(300\,\sqrt{-1}\, e + 24\,\theta\, a\big)\, z^2\zeta^2 w + \big(\tfrac{48}{5}\,\sqrt{-1}\,\theta\, d - 1360\,\sqrt{-1}\, b + 920\, a - \tfrac{48}{5}\,\theta\, c\big)\, z^5\zeta,
\end{aligned}
$$

$$
\begin{aligned}
B^7 \;=\; & \big(\tfrac{1056}{7}\sqrt{-1}\,\theta\, b - 168\,\theta\, a + \tfrac{144}{175}\,\theta^2\, c - \tfrac{144}{175}\sqrt{-1}\,\theta^2\, d\big)\, z^7 \\
& + \big(-48\,\overline{A_{0,0,4}} - 6\,\overline{B_{1,0,3}} + 60\,\theta\, a - 750\sqrt{-1}\, e\big)\, z^3 w^2 + \big(-200\, c\big)\,\zeta w^3 \\
& + \big(\tfrac{56}{25}\,\theta^2\, a - \tfrac{48}{25}\sqrt{-1}\,\theta^2\, b + 1460\, c - 1460\sqrt{-1}\, d + 4\sqrt{-1}\,\theta\, e\big)\, z^6\zeta \\
& + \big(-\tfrac{24}{5}\,\theta^2\, a - 60\sqrt{-1}\,\theta\, e + 100\, c + 100\sqrt{-1}\, d\big)\, z^4\zeta w \\
& + \big(\tfrac{24}{25}\sqrt{-1}\,\theta\, d - \tfrac{24}{25}\,\theta\, c + 900\, a - 900\sqrt{-1}\, b\big)\, z^5\zeta^2 \\
& + \big(360\,\theta\, a + 144\sqrt{-1}\,\theta\, b + 5100\sqrt{-1}\, e\big)\, z^5 w + \big(B_{1,0,3}\big)\, z w^3 \\
& + \big(-1000\, a + 200\sqrt{-1}\, b\big)\, z^3\zeta^2 w + \big(-400\, c + 700\sqrt{-1}\, d\big)\, z^2\zeta w^2,
\end{aligned}
$$

and where :

$$
\begin{aligned}
C^0 &= \sqrt{-1}\, e, \\
C^1 &= \big(2\, a - 2\sqrt{-1}\, b\big)\, z, \\
C^2 &= \big(c - \sqrt{-1}\, d\big)\, z^2 + \big(-2\, c\big)\, w, \\
C^3 &= \big(\tfrac{4}{5}\,\theta\, a + 10\sqrt{-1}\, e\big)\, z w, \\
C^4 &= \big(10\sqrt{-1}\, b\big)\, w^2 + \big(-10\, a - 10\sqrt{-1}\, b\big)\, z^2 w, \\
C^5 &= \big(2\, c - 2\sqrt{-1}\, d\big)\, z^5 + \big(-20\, c + 10\sqrt{-1}\, d\big)\, z w^2, \\
C^6 &= \big(-4\,\theta\, a\big)\, w^3 + \big(2\,\theta\, a - 25\sqrt{-1}\, e\big)\, z^2 w^2, \\
C^7 &= \big(\tfrac{2}{35}\sqrt{-1}\,\theta\, d - \tfrac{2}{35}\,\theta\, c\big)\, z^7 + \big(-20\, a - 20\sqrt{-1}\, b\big)\, z^5 w \\
& \quad + \big(\tfrac{400}{3}\, a + \tfrac{200}{3}\sqrt{-1}\, b\big)\, z w^3, \\
C^8 &= \big(-25\sqrt{-1}\, d + 10\sqrt{-1}\,\theta\, e\big)\, w^4 + \big(-\tfrac{25}{2}\sqrt{-1}\, d + \tfrac{25}{2}\, c\big)\, z^8 + \big(\tfrac{100}{3}\, c + \tfrac{50}{3}\sqrt{-1}\, d\big)\, z^2 w^3 \\
& \quad + \big(10\sqrt{-1}\, d - 10\, c\big)\, z^6 w,
\end{aligned}
$$

$$
\begin{aligned}
C^9 \,&=\, \big(-\tfrac{2}{525}\,\sqrt{-1}\,\theta^2\,d + \tfrac{2}{525}\,\theta^2\,c\big)\,z^9 + \big(\tfrac{4}{7}\,\sqrt{-1}\,\theta\,b + \tfrac{4}{7}\,\theta\,a\big)\,z^7w \\
&\quad + \big(4\,\theta\,a - 50\,\sqrt{-1}\,e\big)\,z^5w^2 + \big(2\,\overline{A_{0,0,4}}\big)\,zw^4,
\end{aligned}
$$

and the related 5 *holomorphic vector fields* e_1*,* e_2*,* e_3*,* e_4*,* e_5 *have structure :*

$$
\begin{aligned}
[e_1, e_2] &= -\tfrac{4}{5}\,\theta\,e_4 - 4\,e_5, & [e_1, e_3] &= 0, & [e_1, e_4] &= 2\,e_2, & [e_1, e_5] &= \tfrac{2}{5}\,\theta\;e_2 - 20\,e_4, \\
& & [e_2, e_3] &= -\,2\,e_2, & [e_2, e_4] &= 0, & [e_2, e_5] &= 0, \\
& & & & [e_3, e_4] &= 2\,e_4, & [e_3, e_5] &= 2\,e_5, \\
& & & & & & [e_4, e_5] &= 0.
\end{aligned}
$$

This Lie algebra $\mathfrak{g}$ has the derived series of dimensions 5, 3, 0, with :

$$
[\mathfrak{g}, \mathfrak{g}] \,=\, \mathsf{Span}\,\big(-\tfrac{4}{5}\,\theta\,e_4 - 4\,e_5,\;\; 2\,e_2,\;\; \tfrac{2}{5}\,\theta\,e_2 - 20\,e_4\big).
$$

These three vector fields form a 3-dimensional Abelian ideal $\mathfrak{a} \subset \mathfrak{g}$, whose value at the origin $0 \in \mathbb{C}^3$ spans a maximally real 3-plane. This is coherent with Fels-Kaup's items **(2a)**, **(2b)**, **(2c)**.

The proofs rely upon studying the *fundamental equation :*

$$
0 \,=\, -\,u' + F'\big(z', \zeta', \overline{z}', \overline{\zeta}', v'\big),
$$

where :

$$\begin{aligned} z' &= f(z, \zeta, w), \\ \zeta' &= g(z, \zeta, w), \\ w' &= h(z, \zeta, w), \end{aligned}$$

for $(z, \zeta, w) \in M$, namely for $u = F(z, \zeta, \overline{z}, \overline{\zeta}, v)$.

These power series normal forms confirm Fels-Kaup's main result that there is a one-to-one correspondence between affine equivalence classes of homogeneous parabolic surfaces $S^2 \subset \mathbb{R}^3$, and biholomorphic equivalence classes of CR homogeneous $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$. In the thickest branch $F_{3,0,0,2,0} = 0 \neq F_{5,0,0,1,0}$, homogeneous models both depend upon a (free) real parameter $\theta \in \mathbb{R}$. Therefore, in this branch, the tubes **(2a)**, **(2b)**, **(2c)** are closed representations of models (a computation of $F_{3,0,0,2,0}$ and of $F_{5,0,0,1,0}$ confirms this). From the implicit equations **(2a)**, **(2b)**, **(2c)**, some (non-closed) graphed power series can be written.

5D PDE Systems Under Fiber-Preserving Equivalences

[Kamran, Kamran-Shadwick, Hsu-Kamran]
[Godlinski, Nurowski]
[Foo-Heyd-M.]
[Heyd]

• **Second order ODEs :**

$$y_{xx} = F(x, y, y_x),$$

under :

$$x' = f(x),$$
$$y' = g(x, y).$$

• **Theorem.** [Foo-M.-Heyd] *There exists a unique choice of formal power series* $(f(x), g(x, y))$ *with :*

$$f(0) = f_x(0) = 0,$$
$$g(0, 0) = g_x(0, 0) = g_y(0, 0) = g_{xy}(0, 0) = 0,$$

such that the map $\varphi(x) = x + f(x)$, $\psi(x,y) = y + g(x,y)$, *brings* $y_{xx} = J(x,y,p)$ *to a normal form :*

$$y_{xx} = \sum_{i,j=0}^{\infty} K_{i,j}(y) x^i p^j,$$

where the $K_{i,j}(y)$ *satisfy :*

(1) $K_{i,0}(y) \equiv 0$ for all $i \geqslant 0$;

(2) $K_{0,1}(y) \equiv 0$;

(3) $K_{i,1}(0) = 0$ for all $i \geqslant 1$;

(4) $K_{0,2}(y) \equiv 0$.

In the real-analytic category, based on a set of principles and guidelines that make full advantage of the formal theory, the convergence problem can be solved in a natural way. Our construction is based on the study of flows of suitable vector fields, with appropriate corrections via the Cauchy-Kovalevskaya theorem. These give our second and the main result, which is the following

• **Theorem** [Foo-M.-Heyd] *There exists a convergent (analytic) fibre-preserving point transformation that sends an analytic second order ordinary differential equation* $y_{xx} = J(x, y, p)$ *to an analytic normal form* $y_{xx} = K(x, y, p)$ *satisfying (1–4).*

As a result, the normal form has the following expansion :

$$
\begin{aligned}
y_{xx} = {} & K_{0,0,3}\frac{p^3}{3!} + K_{1,0,2}\frac{xp^2}{2!} + K_{1,1,1}xyp \\
& + \sum_{i+j+k\geqslant 1} K_{i,j,3+k}\frac{x^i y^j p^{3+k}}{i!j!(3+k)!} + \sum_{i+j+k\geqslant 1} K_{1+i,j,2+k}\frac{x^{1+i} y^j p^{2+k}}{(1+i)!j!(2+k)!} \\
& + \sum_{i+j+k\geqslant 1} K_{1+i,1+j,1+k}\frac{x^{1+i} y^{1+j} p^{1+k}}{(1+i)!(1+j)!(1+k)!},
\end{aligned}
$$

where the coefficients $K_{0,0,3}$, $K_{1,0,2}$, $K_{1,1,1}$ correspond respectively to the values of the relative differential invariants $I_1(K)$, $I_2(K)$, $I_3(K)$ at the origin, which are the only primary ones discovered by Hsu-Kamran.

The problem is the classification of second order ordinary differential equations :

$$Y_{XX} = J(X, Y, Y_X),$$

up to fibre-preserving maps Φ :

$$X = \varphi(x),$$
$$Y = \psi(x, y).$$

This problem was solved by HSU-KAMRAN using method of moving frames, and we briefly summarise the results leading up to the existence of three relative invariants.

Let $J^2(\mathbb{R}, \mathbb{R})$ be the second order Jet space with independent coordinates (x, y, y_x, y_{xx}). For simplicity, we adopt the notation $p := y_x$. The total differential operator D_x along the x-axis is :

$$\mathsf{D}_x := \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + y_{xx}\frac{\partial}{\partial p} + \sum_{j=3}^{+\infty} y_{x^{(k)}} \frac{\partial}{\partial y_{x^{(k-1)}}}.$$

The fibre preserving maps Φ can be prolonged to second order

$$\Phi^2 : \ J^2(\mathbb{R}, \mathbb{R}) \longmapsto J^2(\mathbb{R}, \mathbb{R})$$
$$(x, y, p, y_{xx}) \longmapsto (X, Y, P, Y_{XX})$$

using the recursive formula well-known to Sophus Lie :

$$
\begin{aligned}
X &= \varphi(x), \\
Y &= \psi(x,y), \\
P &= \frac{\mathsf{D}_x Y}{\mathsf{D}_x X} = \frac{\psi_x + \psi_y p}{\varphi_x}, \\
Y_{XX} &= \frac{\mathsf{D}_x P}{\mathsf{D}_x X} = \frac{\varphi_x \psi_{yy} p^2 + 2\varphi_x \psi_{xy} p + \varphi_x \psi_{xx} + (-\psi_x - \psi_y p)\varphi_{xx} + \psi_y \varphi_x y_{xx}}{\varphi_x^3}
\end{aligned}
$$

A given second order differential equation $y_{xx} = K(x,y,p)$ defines a 3-dimensional submanifold $M \subset J^2(\mathbb{R},\mathbb{R})$. The cotangent bundle T^*M is generated by the following 1-forms :

$$
\begin{aligned}
\omega^1 &:= dx, \\
\omega^2 &:= dy - p\, dx, \\
\omega^3 &:= dp - K(x,y,p)\, dx.
\end{aligned}
$$

Now, suppose that Φ^2 sends M to M' given by $Y_{XX} = J(X,Y,P)$. A simple substitution shows that the pull-back Φ^2 transfers $\{\omega^1, \omega^2, \omega^3\}$ to $\{\Omega^1 := dX,\ \Omega^2 := dY - P\, dX,\ \Omega^3 := dP - J(X,Y,P)\, dX\}$ via the initial G-structure :

$$\begin{pmatrix}\Omega^1\\ \Omega^2\\ \Omega^3\end{pmatrix} = \begin{pmatrix}\mathsf{A} & 0 & 0\\ 0 & \mathsf{B} & 0\\ 0 & \mathsf{BC} & \mathsf{B}/\mathsf{A}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix}.$$

By applying CARTAN's method of moving frames, HSU-KAMRAN obtained a 6-dimensional principal bundle $\mathsf{P} \to M$ whose cotangent bundle is generated by six invariant 1-forms $\{\omega^1, \omega^2, \omega^3, \alpha, \beta, \gamma\}$ satisfying the structure equations :

$$\begin{aligned}
d\omega^1 &= \alpha \wedge \omega^1,\\
d\omega^2 &= \beta \wedge \omega^2 + \omega^1 \wedge \omega^3,\\
d\omega^3 &= \gamma \wedge \omega^2 + (\beta - \alpha) \wedge \omega^3,\\
d\alpha &= -2\gamma \wedge \omega^1,\\
d\beta &= \omega^1 \wedge \gamma + I_2\, \omega^1 \wedge \omega^2 - I_1\, \omega^3 \wedge \omega^2,\\
d\gamma &= \gamma \wedge \alpha + I_3\, \omega^1 \wedge \omega^2 + I_2\, \omega^1 \wedge \omega^3.
\end{aligned}$$

Parametrically, there are 3 basic relative invariants :

$$
\begin{aligned}
I_1(K) &= -\frac{\mathsf{A}}{2\mathsf{B}^2} K_{ppp}, \\
I_2(K) &= \frac{1}{2\mathsf{AB}} (\mathsf{D}_x K_{pp} - K_{py}), \\
I_3(K) &= -\mathsf{C} I_2 + \frac{1}{2\mathsf{A}^2\mathsf{B}} (\mathsf{D}_x K_{py} + K_{pp} K_y - K_{py} K_p - 2K_{yy}),
\end{aligned}
$$

while the invariant 1-forms α, β, γ are :

$$
\begin{aligned}
\alpha &= \frac{d\mathsf{A}}{\mathsf{A}} - \left(2\mathsf{C} + \frac{K_p}{\mathsf{A}} \right) \omega^1, \\
\beta &= \frac{d\mathsf{B}}{\mathsf{B}} - \mathsf{C}\omega^1 + \frac{K_{pp}}{2\mathsf{B}} \omega^2, \\
\gamma &= d\mathsf{C} + \mathsf{C}\frac{d\mathsf{A}}{\mathsf{A}} + \left(\frac{K_y}{\mathsf{A}^2} - \frac{\mathsf{C}K_p}{\mathsf{A}} - \mathsf{C}^2 \right) \omega^1 + \left(\frac{K_{py}}{2\mathsf{AB}} - \frac{\mathsf{C}K_{pp}}{2\mathsf{B}} \right) \omega^2 + \frac{K_{pp}}{2\mathsf{B}} \omega^3.
\end{aligned}
$$

• **Theorem.** *Let* $M = \{y_{xx} = 0\}$ *be a second order ordinary differential equation. Then* $\dim \mathfrak{g}_M = 6$*, and its second prolongation* $\mathfrak{g}_M^{(2)}$ *is generated* *by :*

$$
\begin{aligned}
\mathsf{v}_1^{(2)} &= \partial_x, \\
\mathsf{v}_2^{(2)} &= \partial_y, \\
\mathsf{v}_3^{(2)} &= x\partial_y + \partial_p, \\
\mathsf{v}_4^{(2)} &= y\partial_y + p\partial_p + y_{xx}\partial_{y_{xx}}, \\
\mathsf{v}_5^{(2)} &= x\partial_x - p\partial_p - 2y_{xx}\partial_{y_{xx}}, \\
\mathsf{v}_6^{(2)} &= x^2\partial_x + xy\partial_y - (2xp - y)\partial_p - 3xy_{xx}\partial_{y_{xx}}.
\end{aligned}
$$

Proof. By expanding the left-hand side of :

$$
\mathsf{X}^{(2)}(y_{xx})\big|_{y_{xx}=0} \equiv 0,
$$

we obtain :

$$
g_{xx} + (2g_{xy} - f_{xx})p + g_{yy}p^2 = 0.
$$

Solving for $f(x)$, $g(x,y)$ the following system of partial differential equations :

$$\begin{aligned} g_{xx} &= 0, \\ 2g_{xy} &= f_{xx}, \\ g_{yy} &= 0, \end{aligned}$$

the vector field X can be written as a linear combination of the following six vector fields :

$$\begin{aligned} \mathsf{v}_1 &= \partial_x, \\ \mathsf{v}_2 &= \partial_y, \\ \mathsf{v}_3 &= x\partial_y, \\ \mathsf{v}_4 &= y\partial_y, \\ \mathsf{v}_5 &= x\partial_x, \\ \mathsf{v}_6 &= x^2\partial_x + xy\partial_y. \end{aligned}$$

Prolonging all of them to second order finishes the proof. □

The equation :

$$g_{xx} + (2g_{xy} - f_{xx})p + g_{yy}p^2 = 0,$$

is sometimes called the defining function of $\mathfrak{g}_{\{y_{xx}=0\}}$ in $\mathfrak{g}$, and it will also be used to define our normal form. In fact, let $\mathscr{F}_x$ denote the space of all formal power series in x, let $\mathscr{F}_{x,y}$ denote the space of all formal power series in (x, y), and let $\mathscr{F}_{x,y,p}$ denote the space of all formal power series in (x, y, p). Introduce the homological operator :

$$
\begin{aligned}
L: \quad \mathscr{F}_x \times \mathscr{F}_{x,y} &\longrightarrow \mathscr{F}_{x,y,p} \\
(f(x), g(x, y)) &\longmapsto L(f, g) := g_{xx} + (2g_{xy} - f_{xx})p + g_{yy}p^2.
\end{aligned}
$$

We seek a subspace $\mathscr{N} \subset \mathscr{F}_{x,y,p}$ which we call the space of normal forms, satisfying :

$$\mathscr{F}_{x,y,p} = \mathsf{Im}(L) \oplus \mathscr{N},$$

where $\mathscr{N}$ consists of representatives K of classes $[K]$ in $\mathscr{F}_{x,y,p}/\mathsf{Im}(L)$ whose image part has been completely normalised by $L(f, g)$ for some $(f, g) \in \mathscr{F}_x \times \mathscr{F}_{x,y}$ (or in other words, has been completely absorbed into $\mathsf{Im}(L)$).

• **Question.** *How can we find an explicit description of $\mathscr{N}$?*

A clue is to look at the kernel of L. To say that there are two different ways to bring $y_{xx} = J(x, y, p)$ to a normal form is to say that there exist two different (f, g) and $(\widehat{f}, \widehat{g})$ in $\mathscr{F}_x \times \mathscr{F}_{x,y}$ such that :

$$J(x, y, p) = K(x, y, p) + L(f, g),$$
$$J(x, y, p) = K(x, y, p) + L(\widehat{f}, \widehat{g}),$$

with K in $\mathscr{N}$. Then :

$$L(\widehat{f} - f, \widehat{g} - g) = 0,$$

or :

$$(\widehat{f} - f, \widehat{g} - g) \in \mathsf{ker}\ L.$$

Therefore, the choices of normalisations are unique up to *elements in* $\mathsf{ker}\ L$. Fortunately, we have information about this indeterminacy because from a theorem above, $(f(x), g(x, y)) \in \mathsf{ker}\ L$ if and only if the corresponding vector field $\mathsf{X} = f(x)\partial_x + g(x, y)\partial_y$ is a fibre-preserving infinitesimal symmetry of $y_{xx} = 0$, and we have an explicit basis of this Lie algebra $\{\mathsf{v}_1,\ \mathsf{v}_2,\ \mathsf{v}_3,\ \mathsf{v}_4,\ \mathsf{v}_5,\ \mathsf{v}_6\}$.

This piece of information can help us find the subspace $\mathscr{F}' \subset \mathscr{F}_x \times \mathscr{F}_{x,y}$ of the source space on which L is *injective*. Then any element $(f, g) \in \mathscr{F}'$ coming from this subspace will give the *unique* normalisation, hence giving us a precise description of a normal form. To put this idea into action, we first have to find out what $\mathscr{F}'$ is. We expand X in terms of power series :

$$\mathsf{X} = \big(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3+O_x(4)\big)\frac{\partial}{\partial x}+\big(\beta_{0,0}+\beta_{1,0}x+\beta_{0,1}y+\beta_{2,0}x^2+\beta_{1,1}xy$$

Modulo the vector space generated by the first 5 vector fields v_1, v_2, v_3, v_4, v_5 :

$$\mathsf{X} \equiv \big(\alpha_2 x^2+\alpha_3 x^3+O_x(4)\big)\frac{\partial}{\partial x}+\big(\beta_{2,0}x^2+\beta_{1,1}xy+\beta_{0,2}y^2+O_{x,y}(3)\big)\frac{\partial}{\partial y} \ \mathsf{mod}\ \langle \mathsf{v}_1$$

which can be written either as :

$$\mathsf{X} \equiv \big((\alpha_2-\beta_{1,1})x^2+\alpha_3 x^3+O_x(4)\big)\frac{\partial}{\partial x}+\big(\beta_{2,0}x^2+\beta_{0,2}y^2+O_{x,y}(3)\big)\frac{\partial}{\partial y}$$
$$+\beta_{1,1}\mathsf{v}_6 \quad \mathsf{mod}\ \langle \mathsf{v}_1,\mathsf{v}_2,\mathsf{v}_3,\mathsf{v}_4,\mathsf{v}_5\rangle,$$

or :

$$\mathsf{X} \equiv \big(\alpha_3 x^3+O_x(4)\big)\frac{\partial}{\partial x}+\big(\beta_{2,0}x^2+(\beta_{1,1}-\alpha_2)xy+\beta_{0,2}y^2+O_{x,y}(3)\big)\frac{\partial}{\partial y}$$
$$+\alpha_2\mathsf{v}_6 \quad \mathsf{mod}\ \langle \mathsf{v}_1,\mathsf{v}_2,\mathsf{v}_3,\mathsf{v}_4,\mathsf{v}_5\rangle.$$

The two choices are valid, and we will choose the first one for our construction. Hence :

$$\mathsf{X} \equiv \big((\alpha_2 - \beta_{1,1})x^2 + \alpha_3 x^3 + O_x(4)\big)\frac{\partial}{\partial x} + \big(\beta_{2,0}x^2 + \beta_{0,2}y^2 + O_{x,y}(3)\big)\frac{\partial}{\partial y}$$

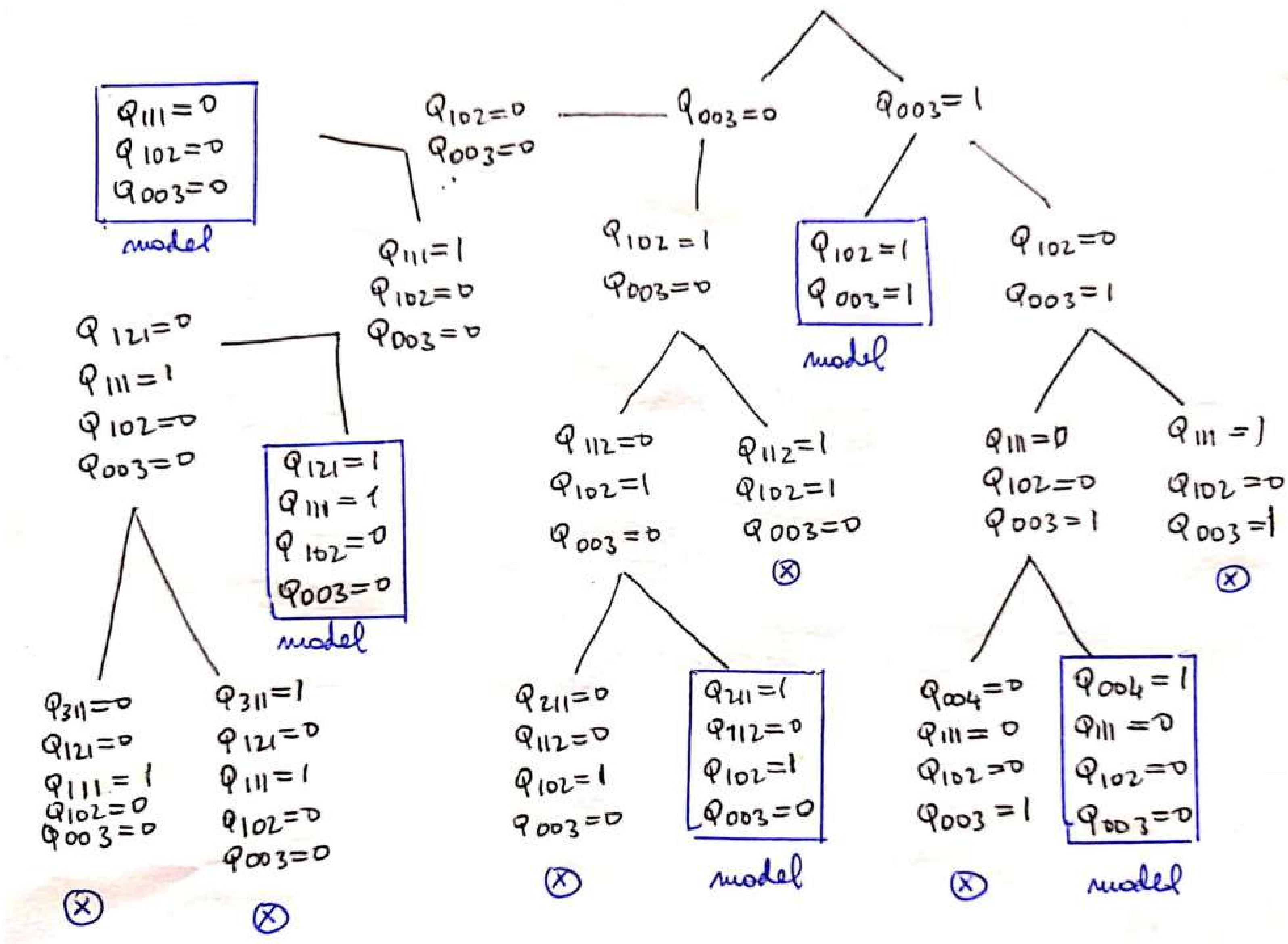
• **Proposition.** *The subspace $\mathscr{F}' \subset \mathscr{F}_x \times \mathscr{F}_{x,y}$ on which the homological operator L is injective is given by the set of all tuples $(f(x), g(x,y))$ satisfying the following conditions :*

(1) $f(x)$ starts with x^2,

(2) $g(x,y)$ starts with homogeneous terms of order 2, with $g_{xy}(0,0) = 0$. □

Homogeneous Models

[Heyd]



The power series method

Assume

$$\{y_{xx} = Q(x, y, y_x)\} \xrightarrow{\varphi^{(2)}} \{y'_{x'x'} = R(x', y', y'_{x'})\},$$

with

$$\varphi : (x, y) \in \mathbb{C}^2 \longmapsto \big(f(x), g(x, y)\big) =: (x', y'),$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = \sum_{i=0}^{n+1} f_i x^i + \mathrm{O}_x(n+2),$$

$$g(x, y) = \sum_{n=0}^{\infty} \sum_{i+j=n} g_{i,j} x^i y^j = \sum_{i+j=0}^{n+2} g_{i,j} x^i y^j + \mathrm{O}_{x,y}(n+3),$$

$n \in \mathbb{N}$.

The power series method

- From the fundamental and vectorial fundamental equations:

$$0 = f_x^3 R\left(f, g, \frac{g_x + g_y p}{f_x}\right) + g_x f_{xx} + g_y f_{xx} p - f_x g_{xx} - 2 f_x g_{xy} p - f_x g_{yy} p^2 - f_x g_y Q(x, y, p),$$

and

$$0 = -B_{xx} + (A_{xx} - 2B_{xy})p + (2A_x - B_y)Q(x, y, p) - B_{yy} p^2 + \\ + A Q_x + B Q_y + \left[B_y + (B_y - A_x)p\right] Q_p,$$

we extract information for normalizing the power series coefficients of Q and R order by order:

- At each order n, by looking at the fundamental equation and by making appropriate choices of coefficients $f_{n+1}, g_{i,j}$ with $i + j = n + 2$, we will normalize the power series coefficients of Q and R.
At the end, such normalizations will end up with normal forms Q^{normal} of Q and with associated diffeomorphisms

$$\{y_{xx} = Q(x, y, y_x)\} \xrightarrow{\varphi_{\square}^{(2)} = \varphi_r^{(2)} \circ \cdots \circ \varphi_1^{(2)}} \{y'_{x'x'} = Q^{\text{normal}}(x', y', y'_{x'})\}.$$

The power series method

Example

For the branching $Q_{0,0,3} = Q_{1,0,2} = 1$,

$$Q^{\mathrm{normal}} = Q \text{ in the theorem.}$$

The power series method

- To find the associated Lie algebra structure, we look the vectorial fundamental equations with

$$\mathcal{L} = \left(\sum_{i=0}^{n+1} A_i x^i + \mathrm{O}_x(n+2) \right) \frac{\partial}{\partial x} + \left(\sum_{i+j=0}^{n+2} B_{i,j} x^i y^j + \mathrm{O}_{x,y}(n+3) \right) \frac{\partial}{\partial y},$$

tangent to

$$\Big\{ y_{xx} = Q(x, y, y_x) \Big\}.$$

- At each order n, by looking at the vectorial fundamental equation, we will normalize coefficients $A_{n+1}, B_{i,j}$ with $i + j = n + 2$, hence we will normalize the coefficients of $\mathcal{L}$.

  At the end of the process, we will obtain vector fields of the form

$$\mathcal{L}^{\mathrm{normal}} = \alpha\Big(A_0, B_{0,0}, B_{1,0}, A_1, A_2, B_{0,1}\Big) \frac{\partial}{\partial x} + \beta\Big(A_0, B_{0,0}, B_{1,0}, A_1, A_2, B_{0,1}\Big) \frac{\partial}{\partial y},$$

with all other coefficients A_* and $B_{*,*}$ normalized, where α and β are linear in their 6 arguments.

The power series method

Next, to obtain the associated Lie algebras in all branches, we will prolong the concerned vector field to $J^1(\mathbb{C}^2_{x,y}) = \mathbb{C}^3_{x,y,p}$, getting

$$\mathcal{L}^{\mathrm{normal},1}.$$

Then putting one of the arguments to 1 and the others to 0, we get generators e_i with $1 \leqslant i \leqslant 3$ or $1 \leqslant i \leqslant 6$ of the associated Lie algebra given in the theorems.
For example :

$$e_1 := \mathcal{L}^{\mathrm{normal},1}(1, 0, ..., 0).$$

The power series method

- Remark : We cannot normalize the coefficients order by order at the infinity.
- $$\pi_{\leqslant n}\left(\sum_{i,j,k} Q_{i,j,k}x^i y^j p^k\right) := \sum_{i+j+k\leqslant n} Q_{i,j,k}x^i y^j p^k \qquad (n\in\mathbb{N}),$$
- In fact, normalization of the coefficients of φ will create a decreasing and stationary sequence of subsets of

$$G_{\mathrm{stab}} = \left\{\varphi \mid \{y_{xx} = Q(x,y,y_x)\} \stackrel{\varphi^{(2)}}{\longrightarrow} \{y'_{x'x'} = R(x',y',y'_{x'})\}\right\}:$$

$$G_{\mathrm{stab}} =: G^0_{\mathrm{stab}} \supset G^1_{\mathrm{stab}} \supset \dots$$

$$\dots \supset G^n_{\mathrm{stab}} =: \left\{\varphi \mid \{y_{xx} = \pi_{\leqslant n}(Q)(x,y,y_x)\} \stackrel{\varphi^{(2)}}{\longrightarrow} \{y'_{x'x'} = \pi_{\leqslant n}(R)(x',y',y'_{x'})\}\right\} \supset \dots$$

$$\dots \supset G^r_{\mathrm{stab}} = G^{r+1}_{\mathrm{stab}} = \dots,$$

- At the order r,the process of normalization stops.
- In geometric terms, we make G_{stab} act on $\{y_{xx} = Q(x,y,y_x)\}$.

The power series method

In the branch

$$Q_{0,0,3} = Q_{1,0,2} = 1.$$

- Assume $Q_{0,0,0} = R_{0,0,0} = 0$
- and φ fixes $(0,0,0)$ in J^1 :

$$f_0 = g_{0,0} = g_{1,0} = 0.$$

- φ is a local diffeo :

$$f_1 \neq 0,\ g_{0,1} \neq 0.$$

- Nouvelle projection :

$$\pi_n\Big(\sum_{i,j,k} Q_{i,j,k} x^i y^j p^k\Big) = \sum_{i+j+k=n} Q_{i,j,k} x^i y^j p^k \qquad (n \in \mathbb{N}),$$

The power series method

Order $n = 0$,

$$Q = \mathrm{O}_{x,y,p}(1) \text{ and } R = \mathrm{O}_{x',y',p'}(1),$$

$$f(x) = \sum_{i=0}^{0+1} f_i x^i + \mathrm{O}_x(2) \text{ and } g(x,y) = \sum_{i+j=0}^{2} g_{i,j} x^i y^j + \mathrm{O}_{x,y}(3).$$

We make G_{stab} act on $\{y_{xx} = Q(x,y,y_x)\}$ on the order 0 by applying π_0 to the fundamental equation :

$$0 = \pi_0 \Big(f_x^3 R + g_x f_{xx} + g_y f_{xx} p - f_x g_{xx} - 2 f_x g_{xy} p - f_x g_{yy} p^2 - f_x g_y Q \Big).$$

Therefore

$$0 = -2 f_1 g_{2,0},$$

hence by $f_1 \neq 0$, we get

$$g_{2,0} = 0.$$

The power series method

Order $n = 1$,

$$Q = \sum_{i+j+k=1} Q_{i,j,k} x^i y^j p^k + \mathrm{O}_{x,y,p}(2) \text{ and } R = \sum_{i+j+k=1} R_{i,j,k} x'^i y'^j p'^k + \mathrm{O}_{x',y',p'}(2),$$

$$f(x) = \sum_{i=0}^{1+1} f_i x^i + \mathrm{O}_x(3) \text{ and } g(x,y) = \sum_{i+j=0}^{3} g_{i,j} x^i y^j + \mathrm{O}_{x,y}(4).$$

We look the action of G_{stab} on $\{y_{xx} = Q(x, y, y_x)\}$ at the order 1 by applying π_1 to the fundamental equation, we get

$$(E1)\ \ 0 = R_{1,0,0} f_1^4 - Q_{1,0,0} f_1 g_{0,1} - 6 f_1 g_{3,0},$$

$$(E2)\ \ 0 = R_{0,1,0} f_1^3 g_{0,1} + R_{0,0,1} f_1^2 g_{1,1} - Q_{0,1,0} f_1 g_{0,1} - 2 f_1 g_{2,1} + 2 f_2 g_{1,1},$$

$$(E3)\ \ 0 = R_{0,0,1} f_1^2 g_{0,1} - Q_{0,0,1} f_1 g_{0,1} - 2 f_1 g_{1,1} + 2 f_2 g_{0,1}.$$

The power series method

$$(E1)\ \ 0 = R_{1,0,0} f_1^4 - Q_{1,0,0} f_1 g_{0,1} - 6 f_1 g_{3,0},$$
$$(E2)\ \ 0 = R_{0,1,0} f_1^3 g_{0,1} + R_{0,0,1} f_1^2 g_{1,1} - Q_{0,1,0} f_1 g_{0,1} - 2 f_1 g_{2,1} + 2 f_2 g_{1,1},$$
$$(E3)\ \ 0 = R_{0,0,1} f_1^2 g_{0,1} - Q_{0,0,1} f_1 g_{0,1} - 2 f_1 g_{1,1} + 2 f_2 g_{0,1}.$$

- By the freeness of $g_{3,0}$ in $(E1)$ we can normalize $R_{1,0,0} = 0$ and by equivalence:
$$Q_{1,0,0} = R_{1,0,0} = 0.$$
So :
$$g_{3,0} = 0.$$
- By looking at $(E3)$, and by a choice of $g_{1,1}$, we can normalize $R_{0,0,1} = 0$ and by equivalence:
$$Q_{0,0,1} = R_{0,0,1} = 0.$$
Therefore $(E3)$ gives
$$g_{1,1} = g_{0,1} \frac{f_2}{f_1}.$$
- By the same idea, from $(E2)$ we get:
$$Q_{0,1,0} = R_{0,1,0} = 0,$$
$$g_{2,1} = g_{0,1} \left(\frac{f_2}{f_1}\right)^2.$$

The power series method

Conclusion :

Lemma

The subgroup $G^1_{\mathrm{stab}} \subset G^0_{\mathrm{stab}}$ which sends $Q = \mathrm{O}_{x,y,p}(2)$ to $R = \mathrm{O}_{x',y',p'}(2)$ consists of fiber-preserving transformations such that

$$f_0 = 0,\ f_1 \neq 0,\ g_{0,0} = g_{1,0} = g_{2,0} = g_{3,0} = 0,\ g_{0,1} \neq 0,$$

$$g_{i,1} = g_{0,1} \left(\frac{f_2}{f_1} \right)^i \qquad (1 \leqslant i \leqslant 2).$$

For order $n \geqslant 2$, it is the same process :
1) Normalization of the $Q_{i,j}$ and $R_{i,j}$.
2) Reducing of G_{stab}.

The power series method

About the vectorial fundamental equation, with

$$\mathcal{L} = \left(\sum_{i=0}^{1+1} A_i x^i + \mathrm{O}_x(n+2) \right) \frac{\partial}{\partial x} + \left(\sum_{i+j=0}^{1+2} B_{i,j} x^i y^j + \mathrm{O}_{x,y}(n+3) \right) \frac{\partial}{\partial y},$$

By applying π_0 to the vectorial fundamental equation, we get:

$$B_{2,0} = 0.$$

Then by π_1, we get

$$B_{3,0} = 0,$$
$$B_{2,1} = 0,$$
$$B_{1,1} = A_2.$$

Hence

$$\mathcal{L} = \left(A_0 + A_1 x + A_2 x^2 + \mathrm{O}_x(n+2) \right) \frac{\partial}{\partial x} + \left(B_{0,0} + B_{1,0} x + B_{0,1} y + \mathrm{O}_{x,y}(n+3) \right) \frac{\partial}{\partial x}$$

About the vectorial fundamental equation, we do the same: at each order n:
Normalization of A_{n+1} and $B_{i,j}$, with $i + j = n + 2$.

5D Fiber-Preserving PDE systems

- **Again :**

$$
\begin{aligned}
z_{xx} &= Q\big(x, y, z, z_x, z_y\big),\\
z_{xy} &= R\big(x, y, z, z_x, z_y\big),\\
z_{yy} &= S\big(x, y, z, z_x, z_y\big),
\end{aligned}
$$

but under fiber-preserving transformations :

$$
\begin{aligned}
x' &= f(x, y),\\
y' &= g(x, y),\\
z' &= h(x, y, z),
\end{aligned}
$$

with :

$$
0 \neq \begin{vmatrix} f_x & f_y & 0 \\ g_x & g_y & 0 \\ h_x & h_y & h_z \end{vmatrix}.
$$

$z = z(x,y)$

$(x, y, z \,;\, a, b, c)$

$z_{xx} = Q(x, y, z, z_x, z_y)$

$z_{xy} = R$

$z_{yy} = S$

$D_x := \frac{\partial}{\partial x} + z_x \frac{\partial}{\partial z} + Q \frac{\partial}{\partial z_x} + R \frac{\partial}{\partial z_y}$

$D_y := \frac{\partial}{\partial y} + z_y \frac{\partial}{\partial z} + R \frac{\partial}{\partial z_x} + S \frac{\partial}{\partial z_y}$

compatibilité

$D_y Q = D_x R$

$D_y R = D_x S$

$x' = f(x,y) = \lambda x + \mu y + \cdots$

$y' = g(x,y) = \nu x + \tau y + \cdots$

$z' = h(x,y,z)$

$0 \neq \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} \lambda & \mu \\ \nu & \tau \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$

$0 \neq h_z$

$z_y(0,0) = 0$

$$z = c + xa + yb + O_{x,y}(2) = F(x,y,a,b,c)$$

$$z' = c' + x'a' + y'b' + O_{x',y'}(2) = F'(x',y',a',b',c')$$

$$z' = R(x,y,z)$$
$$= R\left(f'(x',y'),\ g'(x',y'),\ F\left(f'(x',y'), g'(x',y'), a, b, c\right)\right)$$

$$a' = f'_{x'} R_x + g'_{x'} R_y + \left[f'_{x'} F_x + g'_{x'} F_y\right] R_z$$
$$b' = f'_{y'} R_x + g'_{y'} R_y + \left[f'_{y'} F_x + g'_{y'} F_y\right] R_z$$

$$a' = g'_{x1}(0,0)\, R_x(0,0,\, F(0,0,a,b,c)) + g'_{x1}(0')\, R_y(0,0,\, F(0,0,a,b,c))$$

$$+ \left[g'_{x1}(0)\, F_x(0,0,a,b,c) + g'_{x1}(0)\, F_y(0,0,a,b,c) \right] R_z(0,0,\, F(0,0,a,b,c))$$

$$= \frac{\tau}{\Delta} R_x(0,0,c) - \frac{\nu}{\Delta} R_y(0,0,c) + \left[\frac{\tau}{\Delta} a - \frac{\nu}{\Delta} b \right] R_z(0,0,c)$$

$$a' = \frac{1}{\lambda\tau - \nu\mu} \left\{ \tau R_x(0,0,c) - \nu R_y(0,0,c) + \left[\tau a - \nu b \right] R_z(0,0,c) \right\}$$

$$b' = \frac{1}{\lambda\tau - \nu\mu} \left\{ -\mu R_x(0,0,c) + \lambda R_y(0,0,c) + \left[-\mu a + \lambda b \right] R_z(0,0,c) \right\}$$

$$c' = R(0,0,c)$$

MODÉLE

Rappel 3D: $x(x\partial_x + y\partial_y)$ et $y(x\partial_x + y\partial_y)$ $y\partial_x$ $8-2=6$

5D: $x(x\partial_x + y\partial_y + z\partial_z)$, $y(x\partial_x + y\partial_y + z\partial_z)$, $z(x\partial_x + y\partial_y + z\partial_z)$

$z\partial_x$, $z\partial_y$ $15-3=12$

$z = c + xa + yb$

$e_1 := \partial_x - a\partial_c$

$e_2 := \partial_y - b\partial_c$

$e_3 := \partial_z + \partial_c$

$e_4 := x\partial_z + \partial_a$

$e_5 := y\partial_z + \partial_b$

$e_6 := x\partial_x - a\partial_a$

$e_7 := y\partial_x - a\partial_b$

$e_8 := xy\partial_x + y^2\partial_y + yz\partial_z + c\partial_b$

$e_9 := x\partial_y - b\partial_a$

$e_{10} := y\partial_y - b\partial_b$

$e_{11} := x^2\partial_x + xy\partial_y + xz\partial_z + c\partial_a$

$e_{12} := z\partial_z + a\partial_a + b\partial_b + c\partial_c$

MPL ALG LIE OK

Paramètres d'isotropie:

ordre = 2 $\begin{cases} X_{1,0} & \text{X10} & Y_{1,0} & \text{Y10} & Z_{0,0,1} & \text{Z001} \\ X_{0,1} & \text{X01} & Y_{0,1} & \text{Y01} \end{cases}$

ordre = 3 $\{ X_{1,1}$ X11 $Y_{1,1}$ Y11

Sansintersections:

$h_{0,0,\ell}$ $2\ell=\Theta=2\nu$

$\left\{ \begin{matrix} h_{1,0,\ell} \\ h_{0,1,\ell} \end{matrix} \right\}$ $1+2\ell=\Theta=2\nu+1$

5D: $\acute{E}q[h,i,0,0,\ell]$ $h+i \geq 2$ $F_{h,i,0,0,\ell} := 0 \leftarrow$ $h_{h,i,\ell}$ $h+i+2\ell=\Theta$ Restent

$\acute{E}q[h,i,1,0,0]$ $h+i \geq 3$ $F_{h,i,1,0,0} := 0 \leftarrow$ $f_{h,i}$ Sporadiques Poids 3: $\acute{E}q\,11100$ $F_{11100} := 0 \leftarrow$ f_{11}

$\acute{E}q\,02100$ $F_{02100} := 0 \leftarrow$ f_{02}

$\acute{E}q[h,i,0,1,0]$ $h+i \geq 3$ $F_{h,i,0,1,0} := 0 \leftarrow$ $g_{h,i}$ $\acute{E}q\,20010$ $F_{20010} := 0 \leftarrow$ g_{20}

$\acute{E}q\,11010$ $F_{11010} := 0 \leftarrow$ g_{11}

$\Theta = 2\nu$: $\acute{E}q[2,0,2,0,\ell]$ $\ell \geq 0$ $F_{2,0,2,0,\ell} := 0 \leftarrow$ $h_{0,0,\ell+2}$

$F_{1,1,1,1,\ell} := 0$

$\Theta = 2\nu+1$: $\left\{ \begin{matrix} \acute{E}q[2,0,1,0,\ell] & \ell \geq 0 & F_{2,0,1,0,\ell} := 0 \leftarrow h_{1,0,\ell+1} \\ \acute{E}q[0,2,0,1,\ell] & \ell \geq 0 & F_{0,2,0,1,\ell} := 0 \leftarrow h_{0,1,\ell+1} \end{matrix} \right.$

ISOTROPIE = LIBERTÉ NON CONSOMMÉE

$f_{1,0}$ $f_{0,1}$ $h_{0,0,1}$ $f_{2,0}$

$g_{1,0}$ $g_{0,1}$ $g_{0,2}$

TODO NEXT: n_f jusqu'à l'ordre 4 avec paramètres ~~présents~~ isotropiques

f_{10} α f_{01} β h_{001} χ f_{20} φ

g_{10} γ g_{01} δ g_{02} ψ

$A_{hiil} = 0 \quad h_{ti}+2\ell \leq 17$

$h_{ti} \geq 3$

$h_{ti} = 3 \quad h_{ti} = 2$

$F_{h,i,j,1,0,\ell}$

$h_{ti} = 2$

Kerbourch [illegible] ← 04 68 80 35 82 n^2+1+n

03 81 88 77 09

$-\gamma' + b' + \delta' \ldots$

EHP 5014

$$F = c + xa + yb + F_{11200}\, xya^2 + F_{02200}\, y^2a^2 + F_{20110}\, x^2ab + F_{11110}\, xyab + F_{02110}\, y^2ab + F_{20020}\, x^2b^2 + F_{11020}\, xyb^2 + F_{02020}\, y^2b^2 + o(5)$$

Simpler.

$$F = \sum_{i+j+k+\ell+2m \leq \theta} F_{ijk\ell m}\, x^i y^j a^k b^\ell c^m \qquad 0 \leq v \leq \theta$$

02020

Lexicog.: $F = c + xa + yb + F_{20110}\, x^2ab + F_{20020}\, x^2b^2 + F_{11200}\, xya^2 + F_{11110}\, xyab + F_{11020}\, xyb^2 + F_{02200}\, y^2a^2 + F_{02110}\, y^2ab + F_{02020}\, y^2b^2$

$G = c' + x'a' + y'b' + G_{20110}\, x'^2a'b' + G_{20020}\, x'^2b'^2 + G_{11200}\, x'y'a'^2 + G_{11110}\, x'y'a'b' + G_{11020}\, x'y'b'^2 + G_{02200}\, y'^2a'^2 + G_{02110}\, y'^2a'b' + G_{02020}\, y'^2b'^2$

$x' = \alpha x + \beta y + \cdots$

$y' = \gamma x + \delta y + \cdots$

$z' = \gamma z + \cdots$ (1)

$a' = \frac{\delta}{\Delta} a - \frac{\gamma}{\Delta} b + \cdots$

$b' = -\frac{\beta}{\Delta} a + \frac{\alpha}{\Delta} b + \cdots$ ← Matrice inverse transposée

$c' = c + \cdots$

$x = 1$ pour simplifier $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = [\ldots$

Eqfond: $0 = -z' + F'$

CCL : Action sur $\mathbb{C}^8$ de $GL(2,\mathbb{C})$ OPS $\alpha\delta - \gamma\beta = 1$

$z = c + xa + yb +$ ~~[crossed out]~~

$x^2 \quad + 0 \quad + F_{20110}\, x^2 ab + F_{20020}\, x^2 b^2$

$xy \quad + F_{11200}\, xya^2 + F_{11110}\, xyab + F_{11020}\, xyb^2$

$y^2 \quad + F_{02200}\, y^2 a^2 + F_{02110}\, y^2 ab + F_{02020}\, y^2 b^2$

$z' = c' + x'a' + y'b'$

$x'^2 \quad + 0 \quad + G_{20110}\, x'^2 a' b' + G_{20020}\, x'^2 b'^2$

$x'y' \quad + G_{11200}\, x'y'a'^2 + G_{11110}\, x'y'a'b' + G_{11020}\, x'y'b'^2$

$y'^2 \quad + G_{02200}\, y'^2 a'^2 + G_{02110}\, y'^2 a'b' + G_{02020}\, y'^2 b'^2$

$\forall \chi \in \mathbb{C}^*$

PRP: $\forall \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL_2(\mathbb{C})$

$z = xa + yb + O(3)$

$z' = x'a' + y'b' + O(3)$

$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \frac{1}{\alpha\delta - \gamma\beta} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$

__Action :__ $\Delta := \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \gamma\beta$

$x' = \alpha x + \beta y + \cdots$

$y' = \gamma x + \delta y + \cdots$

$z' = \chi z + \cdots$

$a' = \left(\frac{\delta}{\Delta} a - \frac{\gamma}{\Delta} b\right)\chi + \cdots$

$b' = \left(-\frac{\beta}{\Delta} a + \frac{\alpha}{\Delta} b\right)\chi + \cdots$

d'après la page 6

$c' = \chi c + \cdots$

$\downarrow \frac{1}{(\alpha\delta-\beta\gamma)^2}\times$ always in factor, for the eight coeffs

Les F en fonction des G ne sont pas plus simples, donc on admet qu'il y a une inversion implicite à effectuer par le lecteur

$$G_{20200} = \Big[(\alpha^2\delta^2 + 2\alpha\beta\gamma\delta) F_{20200} + (2\alpha\beta\delta^2 + \beta^2\delta\gamma) F_{20110} + 3\beta^2\delta^2 F_{20020}$$
$$+ (-2\alpha^2\delta\gamma - \alpha\beta\gamma^2) F_{11200} + (-\alpha\beta\delta^2 - 2\beta^2\gamma\delta) F_{11020}$$
$$+ 3\alpha^2\gamma^2 F_{0220} + (\alpha^2\gamma\delta + 2\alpha\beta\gamma^2) F_{02110} + (2\alpha\beta\delta\gamma + \beta^2\gamma^2) F_{02020}\Big]$$

$$G_{20110} = 2\alpha\gamma\delta^2 F_{20200} + (\alpha\delta^3 + \beta\gamma\delta^2) F_{20110} + 2\beta\delta^3 F_{20020}$$
$$- 2\alpha\gamma^2\delta F_{11200} - 2\beta\gamma\delta^2 F_{11020}$$
$$+ 2\alpha\gamma^3 F_{02200} + (\alpha\gamma^2\delta + \beta\gamma^3) F_{02110} + 2\beta\gamma^2\delta F_{02020}$$

$$G_{20020} = \gamma^2\delta^2 F_{20200} + \gamma\delta^3 F_{20110} + \delta^4 F_{20020}$$
$$- \gamma^3\delta F_{11200} - \gamma\delta^3 F_{11020}$$
$$+ \gamma^4 F_{02200} + \gamma^3\delta F_{02110} + \gamma^2\delta^2 F_{02020}$$

$$G_{11200} = -2\alpha^2\beta\delta F_{20200} - 2\alpha\beta^2\delta F_{20110} - 2\beta^3\delta F_{20020}$$
$$+ (\alpha^3\delta + \alpha^2\beta\gamma) F_{11200} + (\alpha\beta^2\delta + \beta^3\gamma) F_{11020}$$
$$- 2\alpha^3\gamma F_{02200} - 2\alpha^2\beta\gamma F_{02110} - 2\alpha\beta^2\gamma F_{02020}$$

$$G_{11020} = -2\beta\delta\gamma^2 F_{20200} - 2\beta\gamma\delta^2 F_{20110} - 2\beta\delta^3 F_{20020}$$
$$+ (\alpha\gamma^2\delta + \beta\gamma^3) F_{11200} + (\alpha\delta^3 + \beta\gamma\delta^2) F_{11020}$$
$$- 2\alpha\gamma^3 F_{02200} - 2\alpha\gamma^2\delta F_{02110} - 2\alpha\gamma\delta^2 F_{02020}$$

Actions: □ Dilatation $G_{20200} := \underbrace{\frac{1}{\chi(\alpha\delta-\beta\gamma)^2}}_{=:\,\sigma\,\in\,\mathbb{C}^*} F_{20200}$, etc., $G_{02020} := \underbrace{\frac{1}{\chi(\alpha\delta-\beta\gamma)^2}}_{=:\,\sigma\,\in\,\mathbb{C}^*} F_{02020}$

□ "Rotation", action de $GL(2,\mathbb{C}) \ni \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ avec $0 \neq \alpha\delta-\beta\gamma$ sur $\mathbb{C}^8$:

$$\begin{pmatrix} G_{20200} \\ G_{20110} \\ G_{20020} \\ G_{11200} \\ G_{11020} \\ G_{02200} \\ G_{02110} \\ G_{02020} \end{pmatrix} = \begin{pmatrix}
\alpha\delta(\alpha\delta+2\beta\gamma & \beta\delta(2\alpha\delta+\beta\gamma) & 3\beta^2\delta^2 & -\alpha\gamma(2\alpha\delta+\beta\gamma) & -\beta\delta(\alpha\delta+2\beta\gamma) & 3\alpha^2\gamma^2 & \alpha\gamma(\alpha\delta+2\beta\gamma) & \beta\gamma(2\alpha\delta+\beta\gamma) \\
2\alpha\gamma\delta^2 & \delta^2(\alpha\delta+\beta\gamma) & 2\beta\delta^3 & -2\alpha\gamma^2\delta & -2\beta\gamma\delta^2 & 2\alpha\gamma^3 & \gamma^2(\alpha\delta+\beta\gamma) & 2\beta\gamma^2\delta \\
\delta^2\gamma^2 & \gamma\delta^3 & \delta^4 & -\gamma^3\delta & -\gamma\delta^3 & \gamma^4 & \gamma^3\delta & \gamma^2\delta^2 \\
-2\alpha^2\beta\delta & -2\alpha\beta^2\delta & -2\beta^3\delta & \alpha^2(\alpha\delta+\beta\gamma) & \beta^2(\alpha\delta+\beta\gamma) & -2\alpha^3\gamma & -2\alpha^2\beta\gamma & -2\alpha\beta^2\gamma \\
-2\beta\gamma^2\delta & -2\beta\gamma\delta^2 & -2\beta\delta^3 & \gamma^2(\alpha\delta+\beta\gamma) & \delta^2(\alpha\delta+\beta\gamma) & -2\alpha\gamma^3 & -2\alpha\gamma^2\delta & -2\alpha\gamma\delta^2 \\
\alpha^2\beta^2 & \alpha\beta^3 & \beta^4 & -\alpha^3\beta & -\alpha\beta^3 & \alpha^4 & \alpha^3\beta & \alpha^2\beta^2 \\
2\alpha\beta^2\gamma & \beta^2(\alpha\delta+\beta\gamma) & 2\beta^3\delta & -2\alpha^2\beta\gamma & -2\alpha\beta^2\delta & 2\alpha^3\gamma & \alpha^2(\alpha\delta+\beta\gamma) & 2\alpha^2\beta\delta \\
\beta\gamma(2\alpha\delta+\beta\gamma) & \beta\delta(\alpha\delta+2\beta\gamma) & 3\beta^2\delta^2 & -\alpha\gamma(\alpha\delta+2\beta\gamma) & -\beta\delta(2\alpha\delta+\beta\gamma) & 3\alpha^2\gamma^2 & \alpha\gamma(2\alpha\delta+\beta\gamma) & \alpha\delta(\alpha\delta+2\beta\gamma)
\end{pmatrix} \begin{pmatrix} F_{20200} \\ F_{20110} \\ F_{20020} \\ F_{11200} \\ F_{11020} \\ F_{02200} \\ F_{02110} \\ F_{02020} \end{pmatrix}$$

8 coeffs $F_{20200}, \ldots, F_{02020}$ $\xrightarrow[\mathbb{C}^5]{\text{action } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, x}$ 8 coeffs $G_{20200}, \ldots, G_{02020}$

norm $F_{11110} = 0$ $G_{11110} = 0$

Matrice A [on a enlevé x_1 $(\alpha\delta - \beta\gamma)^*$]

vraie: $\frac{1}{(\alpha\delta-\beta\gamma)^2 x}$ A

$$\begin{pmatrix} K_1 \\ \vdots \\ K_8 \end{pmatrix} = \begin{pmatrix} P^{-1} \end{pmatrix}_{8\times 8} \begin{pmatrix} F_{20200} \\ \vdots \\ F_{02020} \end{pmatrix} \qquad \begin{pmatrix} L_1 \\ \vdots \\ L_8 \end{pmatrix} = \begin{pmatrix} P^{-1} \end{pmatrix} \begin{pmatrix} G_{20200} \\ \vdots \\ G_{02020} \end{pmatrix}$$

nouveaux

$$A_g := P^{-1} A P \qquad \left(P^{-1} \to \frac{1}{(\alpha\delta-\beta\gamma)^2 x} \right)$$

$$\begin{pmatrix} L_1 \\ \vdots \\ L_8 \end{pmatrix} = \begin{pmatrix} 5\times 5_{\alpha,\beta,\gamma,\delta,x} & 0 \\ 0 & 3\times 3 \end{pmatrix} \begin{pmatrix} K_1 \\ \vdots \\ K_8 \end{pmatrix}$$

A_g ↗ (the matrix)

$0 \to -2$

G stab $I, \dots, VI$

3×3: $(\alpha\delta-\beta\gamma) 1$

.... G stab finaux

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{pmatrix} = \begin{pmatrix} B_{11}(\alpha,\beta,\gamma,\delta,\underline{x}) & & \\ & \ddots & \\ & & B_{55} \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_5 \\ K_8 \end{pmatrix}$$

$$
B = \frac{1}{\omega}\begin{bmatrix}
\delta^4 & \delta^3\gamma & \frac{1}{2}\delta^2\gamma^2 & \frac{1}{6}\gamma^3\delta & \frac{1}{24}\gamma^4 & 0 & 0 & 0 \\
4\beta\delta^3 & \delta^2(\alpha\delta+3\beta\gamma) & \delta\gamma(\alpha\delta+\beta\gamma) & \frac{1}{6}\gamma^2(3\alpha\delta+\beta\gamma) & \frac{1}{6}\alpha\gamma^3 & 0 & 0 & 0 \\
12\beta^2\delta^2 & 6\beta\delta(\alpha\delta+\beta\gamma) & \alpha^2\delta^2+4\alpha\delta\beta\gamma+\beta^2\gamma^2 & \alpha\gamma(\alpha\delta+\beta\gamma) & \frac{1}{2}\alpha^2\gamma^2 & 0 & 0 & 0 \\
24\beta^3\delta & 6\beta^2(3\alpha\delta+\beta\gamma) & 6\alpha\beta(\alpha\delta+\beta\gamma) & \alpha^2(\alpha\delta+3\beta\gamma) & \alpha^3\gamma & 0 & 0 & 0 \\
24\beta^4 & 24\alpha\beta^3 & 12\beta^2\alpha^2 & 4\alpha^3\beta & \alpha^4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta^2(\alpha\delta-\beta\gamma) & 2\delta\gamma(\alpha\delta-\beta\gamma) & -\gamma^2(\alpha\delta-\beta\gamma) \\
0 & 0 & 0 & 0 & 0 & \beta\delta(\alpha\delta-\beta\gamma) & (\alpha\delta-\beta\gamma)(\alpha\delta+\beta\gamma) & -\alpha\gamma(\alpha\delta-\beta\gamma) \\
0 & 0 & 0 & 0 & 0 & -\beta^2(\alpha\delta-\beta\gamma) & -2\alpha\beta(\alpha\delta-\beta\gamma) & \alpha^2(\alpha\delta-\beta\gamma)
\end{bmatrix}
$$

Since homothetie let invariant the invariant spaces

$$
\rho : \begin{cases} GL_2(\mathbb{C}) \longrightarrow GL_8(\mathbb{C}) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto B \end{cases}.
$$

Now, we will connect this problem with the binar forms equivalence.

we consider equivalences of binary forms:

$$Q(x,y) = \sum_{i=0}^{n} \binom{n}{i} a_i x^{n-i} y^{i} \qquad (\mathbb{C}[x,y],\ n \geq 0),$$

under the effect of invertible linear changes of variables

$$(\otimes) \quad \begin{cases} u = \alpha x + \beta y, \\ v = \gamma x + \delta y, \\ \alpha\delta - \beta\gamma \neq 0. \end{cases}$$

Under a such linear transformation, the polynomial $Q(x,y)$ is mapped to a new polynomial $R(u,V)$, defined so that

$$(\circledast)' \qquad R(u,v) = R(\alpha x + \beta y, \gamma x + \delta y) = Q(x,y).$$

Theorem (Olver, Classical Invariant Theory)

Let $Q(x,y)$ and $R(u,v)$ be two binary forms related by the linear transformation ⊗. Then their coefficients are related by the explicit formulae

$$a_i = \sum_{k=0}^{n} b_k \left\{ \sum_{j=\max\{0,\, i+k-n\}}^{\min\{i,k\}} \binom{i}{j}\binom{n-i}{k-j} \alpha^{j} \beta^{k-j} \gamma^{i-j} \delta^{n+j-i-k} \right\} \qquad (0 \le i \le n),$$

where

$$Q(x,y) = \sum_{i=0}^{n} \binom{n}{i} a_i x^{n-i} y^{i},$$

$$R(x,y) = \sum_{j=0}^{n} \binom{n}{j} b_j x^{n-j} y^{j}.$$

Case $n=2$

Let Q and R be two binary forms of degree $n=2$, so that

$$Q(x,y) = a_0 x^2 + 2a_1 xy + a_2 y^2,$$

$$R(u,v) = b_0 u^2 + 2b_1 uv + b_2 v^2,$$

satisfying $\circledast$ and $\circledast'$.

By computation or by the theorem, we find

$$b_0 = a_0\alpha^2 + 2a_1\alpha\gamma + a_2\gamma^2,$$

$$b_1 = a_0\alpha\beta + a_1\alpha\delta + a_1\beta\gamma + a_2\delta\gamma,$$

$$b_2 = a_0\beta^2 + 2a_1\beta\delta + a_2\delta^2.$$

It is equivalent to

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta+\beta\gamma & \delta\gamma \\ \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} =: M_2 \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Case $n = 4$

Let Q and R be two binary forms of degree $n = 4$, so that

$$Q(x,y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4,$$

$$R(u,v) = b_0 u^4 + 4b_1 u^3 v + 6b_2 u^2 v^2 + 4b_3 u v^3 + b_4 v^4,$$

satisfying $\circledast$ and $\circledast$.

By computation, we find

$$b_0 = a_0 \alpha^4 + 4a_1 \alpha^3 \gamma + 6a_2 \alpha^2 \gamma^2 + 4a_3 \alpha \gamma^3 + a_4 \gamma^4,$$

$$b_1 = a_0 \alpha^3 \beta + a_1 \alpha^3 \delta + 3a_1 \alpha^2 \beta \gamma + 3a_2 \alpha^2 \delta \gamma + 3a_2 \alpha \beta \gamma^2 + 3a_3 \alpha \delta \gamma^2 +$$
$$+ a_3 \beta \gamma^3 + a_4 \delta \gamma^3,$$

$$b_2 = a_0 \alpha^2 \beta^2 + 2a_1 \alpha^2 \beta \delta + 2a_1 \alpha \beta^2 \gamma + a_2 \alpha^2 \delta^2 + 4a_2 \alpha \beta \delta \gamma + a_2 \beta^2 \gamma^2 + 2a_3 \alpha \delta^2 \gamma +$$
$$+ 2a_3 \beta \delta \gamma^2 + a_4 \delta^2 \gamma^2,$$

$$b_3 = a_0 \alpha \beta^3 + 3a_1 \alpha \beta^2 \delta + a_1 \beta^3 \gamma + 3a_2 \alpha \beta \delta^2 + 3a_2 \beta^2 \delta \gamma + a_3 \alpha \delta^3 +$$
$$+ 3a_3 \beta \delta^2 \gamma + a_4 \delta^3 \gamma,$$

$$b_4 = a_0 \beta^4 + 4a_1 \beta^3 \delta + 6a_2 \beta^2 \delta^2 + 4a_3 \beta \delta^3 + a_4 \delta^4.$$

[Olver 1999]

Canonical Forms for Complex Binary Quartics			
I.	$p^4 + \mu p^2 + 1$	$\mu \neq \pm 2,\ \Delta \neq 0$	simple roots
II.	$p^2 + 1$	$\Delta = 0,\ T \not\equiv 0$	one double root
III.	p^2	$\Delta = 0,\ T \equiv 0,\ i \neq 0$	two double roots
IV.	p	$i = j = 0,\ H \not\equiv 0$	triple root
V.	1	$H \equiv 0,\ Q \neq 0$	quadruple root
VI.	0	$Q \equiv 0$	

[Gurevich 1964]

Type	Canonical form	Arithmetic characteristics $(\rho\rho_h\sigma)$	Algebraic characteristics
I	$x^4+6\mu x^2y^2+y^4$, $\mu \neq \pm\frac{1}{3}$	(222)	$i^3-27j^2 \neq 0$
II	$6x^2y^2+y^4$	(221)	$i^3-27j^2=0$, $2iH-3jf\neq 0$
III	$6x^2y^2$	(220)	$2iH-3jf=0$, $i\neq 0$
IV	$4x^3y$	(210)	$i=j=0$, $H\neq 0$
V	x^4	(100)	$H=0$, $j\neq 0$
VI	the form vanishes identically	(000)	$f=0$

• **Open problem.** *Classify homogeneous PDEs* $z_{xx}=Q$, $z_{xy}=R$, $z_{yy}=S$ *over* $\mathbb{R}$. [$\mathbb{C}$: Doubrov-Medvedev-The]

12. If the coefficients of a form f of Type I are real, prove that the form H decomposes into real linear factors for at least one of the roots of the resolvent (25

13. Classify binary fourth order forms in the real domain, assuming that the criminant $i^3-27j^2\neq 0$.

14. Classify in the real domain those binary fourth order forms for which the criminant i^3-27j^2 is equal to zero.

12. Use *11*, (25.19) and (25.26); consider separately the cases $i^3-27j^2 > 0$ and $i^3-27j^2 < 0$.

13. ***Hint***: Use *10* and *12*; apply Descartes' rule to (25.27).

Answer: I^a. Canonical form: $x^4+6\mu x^2y^2+y^4$, $\mu < -\frac{1}{3}$. The form determines four real points. Algebraic characteristic: $i^3-27j^2 > 0$; for any point of the straight line $H < 0$; $12H^2-if^2 > 0$.

I^b, I^c. Canonical forms: $\alpha(x^4+6\mu x^2y^2+y^4)$, $\mu > -\frac{1}{3}$, $\alpha = \pm 1$. The form determines four imaginary points. Algebraic characteristic: $i^3-27j^2 > 0$; for any point of the straight line $\alpha f > 0$ and either $H \geqq 0$ or $12H^2-if^2 < 0$.

I^d. Canonical form: $x^4+6\mu x^2y^2-y^4$. Among the points determined by the form, two are real and two imaginary. Algebraic characteristic: $i^3-27j^2 < 0$.

14. ***Hint***: Cf. (25.26) and *11*.

Answer: Types IV and VI are preserved. Type II becomes four with the canonical forms $6\alpha x^2y^2+\beta y^4$, $\alpha = \pm 1$, $\beta = \pm 1$, $\alpha j < 0$, $\alpha\beta(2iH-3jf) \geqq 0$. Type III becomes four: III^a, III^b (for $H \leqq 0$) with the canonical forms $6\alpha x^2y^2$, $\alpha j < 0$ and III^c, III^d (for $H \geqq 0$) with the canonical forms $\gamma(x^2+y^2)^2$, $\gamma = \pm 1$, $\gamma j > 0$. Type V becomes two with canonical forms αx^4, $\alpha = \pm 1$.

Next, we use the classification of binary forms of Olver (Classical Invariant Theory)

<u>Theorem</u> Every binary form $Q = a_0x^2 + 2a_1xy + a_2y^2$ of order $n=2$ under invertible linear changes of variables is, in a unique way, equivalent to

case I : xy $\Delta \neq 0$,

case II : y^2 $\Delta = 0$,

case III : 0 $Q \equiv 0$,

where Δ is the discriminant of Q.

$= a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$

Theorem Every binary form Q of order $n=4$ under invertible linear changes of variables is, in a unique way, equivalent to

case I:	$x^4+\mu x^2y^2+y^4$	$\mu \neq \pm 2$, $\Delta \neq 0$,
case II:	$x^2y^2+y^4$	$\Delta = 0$, $T \not\equiv 0$,
case III:	x^2y^2	$\Delta = 0$, $T \equiv 0$, $i \neq 0$,
case IV:	xy^3	$i=j=0$, $H \not\equiv 0$,
case V:	y^4	$H \equiv 0$, $Q \not\equiv 0$
case VI:	0	$Q \equiv 0$,

where Δ is the discriminant of Q,

$i = a_0a_4 - 4a_1a_3 + 3a_2^2$,

$$j = \det\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix},$$

Summary

Sans binomiaux : $P = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} = P^{-1} \qquad F' = P \cdot F$

$$\begin{pmatrix} G_{02200} \\ G_{02110} \\ \underset{\cdot}{-} G_{11110} \\ G_{20110} \\ G_{20020} \end{pmatrix} = \frac{1}{x(\alpha\delta-\beta\gamma)^2} \begin{pmatrix} \alpha^4 & \alpha^3\beta & \alpha^2\beta^2 & \alpha\beta^3 & \beta^4 \\ 4\alpha^3\gamma & \alpha^2(\alpha\delta+3\beta\gamma) & 2\alpha\beta(\alpha\delta+\beta\gamma) & \beta^2(3\alpha\delta+\beta\gamma) & 4\beta^3\delta \\ 6\alpha^2\gamma^2 & 3\alpha\gamma(\alpha\delta+\beta\gamma) & \alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2 & 3\beta\delta(\alpha\delta+\beta\gamma) & 6\beta^2\delta^2 \\ 4\alpha\gamma^3 & \gamma^2(3\alpha\delta+\beta\gamma) & 2\delta\gamma(\alpha\delta+\beta\gamma) & \delta^2(\alpha\delta+3\beta\gamma) & 4\beta\delta^3 \\ \gamma^4 & \delta\gamma^3 & \delta^2\gamma^2 & \gamma\delta^3 & \delta^4 \end{pmatrix} \begin{pmatrix} F_{02200} \\ F_{02110} \\ \underset{\cdot}{-} F_{11110} \\ F_{20110} \\ F_{20020} \end{pmatrix}$$

Six formes normales pour les coeffs :

	F_{02200}	F_{02110}	$\underset{\cdot}{-} F_{11110}$	F_{20110}	F_{20020}	Racines Multiplicités :
I	1	0	$\mu \neq \pm 2$	0	1	1, 1, 1, 1
II	0	0	1	0	1	2, 1, 1
III	0	0	1	0	0	2, 2
IV	0	0	0	1	0	3, 1
V	0	0	0	0	1	4
VI	0	0	0	0	0	
	a_0	a_1	a_2	a_3	a_4	

Forme binaire sans binomiaux :

$a_0 X^4 + a_1 X^3 Y + a_2 X^2 Y^2 + a_3 X Y^3 + a_4 Y^4$

cas II : $X^2Y^2 + Y^4$ alors $X \mapsto \sqrt{-1}\, X$ change signe, donc $\underset{\cdot}{-} F_{11110}$ neutralisé

cas III : pareille obs !

Action ing^{ale}; forme quartique sans binomiaux: GL_2:

$$A := \begin{pmatrix} a^4 & a^3c & a^2c^2 & ac^3 & c^4 \\ 4a^3b & a^2(ad+3bc) & 2ac(ad+bc) & c^2(3ad+bc) & 4c^3d \\ 6a^2b^2 & 3ab(ad+bc) & a^2d^2+4abcd+b^2c^2 & 3cd(ad+bc) & 6c^2d^2 \\ 4ab^3 & b^2(3ad+bc) & 2bd(ad+bc) & d^2(ad+3bc) & 4cd^3 \\ b^4 & b^3d & b^2c^2 & bd^3 & d^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} =: \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

ing^{al}:

$$\underline{AL} := \begin{array}{c|cccc} & La & Lb & Lc & Ld \\ \hline \partial a_0 & 4a_0 & 0 & a_1 & 0 \\ \partial a_1 & 3a_1 & 4a_0 & 2a_2 & a_1 \\ \partial a_2 & 2a_2 & 3a_1 & 3a_3 & 2a_2 \\ \partial a_3 & a_3 & 2a_2 & 4a_4 & 3a_3 \\ \partial a_4 & 0 & a_3 & 0 & 4a_4 \end{array}$$

$\Delta_{1234} = -128a_0^2a_2a_4 + 144a_0^2a_3^2 + 48a_0a_1^2a_4 - 112a_0a_1a_2a_3 + 32a_0a_2^3 + 24a_1^3a_3 - 8a_1^2a_2^2$

$\Delta_{1235} = 192a_0^2a_3a_4 - 128a_0a_1a_2a_4 - 12a_0a_1a_3^2 + 16a_0a_2^2a_3 + 36a_1^3a_4 - 4a_1^2a_2a_3$

$\Delta_{1245} = 256a_0^2a_4^2 - 32a_0a_1a_3a_4 - 64a_0a_2^2a_4 + 24a_0a_2a_3^2 + 24a_1^2a_2a_4 - 8a_1^2a_3^2$ — gb ne donne rien...

$\Delta_{1345} = 192a_0a_1a_4^2 - 128a_0a_2a_3a_4 + 36a_0a_3^3 - 12a_1^2a_3a_4 + 16a_1a_2^2a_4 - 4a_1a_2a_3^2$

$\Delta_{2345} = -128a_0a_2a_4^2 + 48a_0a_3^2a_4 + 144a_1^2a_4^2 - 112a_1a_2a_3a_4 + 24a_1a_3^3 + 32a_2^3a_4 - 8a_2^2a_3^2$

$-28-2/3-$

Exceptions: I: $x^4 - 2x^2y^2 + y^4 = (x-y)^2(x+y)^2 \Leftrightarrow$
$x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2 = (x-iy)^2(x+iy)^2$

obs: $x^4 + \mu x^2y^2 + y^4 \Updownarrow x^4 - \mu x^2y^2 + y^4$ via $y \mapsto iy$

$\Delta_{1234} = 32\mu(\mu-2)(\mu+2)$
$\Delta_{1235} = 0$
$\Delta_{1245} = -64(\mu-2)(\mu+2)$
$\Delta_{1345} = 0$
$\Delta_{2345} = 32\mu(\mu-2)(\mu+2)$

$rg = 4 \Leftrightarrow \mu \neq \pm 2$

$$AL = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 2\mu & 0 \\ 2\mu & 0 & 0 & 2\mu \\ 0 & 2\mu & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

II: $x^2y^2 + y^4 = y^2(x^2+y^2)$ 2,1,1

$$AL = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad rg = 4$$

$\Delta_{1234} = 0$
$\Delta_{1235} = 0$
$\Delta_{1245} = 0$
$\Delta_{1345} = 0$
$\Delta_{2345} = 32$

III: x^2y^2 2,2

$$AL = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad rg = 3$$

IV: xy^3

$$AL = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$rg = 3$

V: y^4

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$rg = 2$

Réordonner :

$$\begin{pmatrix} G_{02200} \\ G_{02110} \\ G_{1110} \\ G_{20110} \\ G_{20020} \end{pmatrix} = \frac{1}{\chi(\alpha\delta-\beta\gamma)^2} \underbrace{\begin{pmatrix} \alpha^4 & \alpha^3\beta & \mp\alpha^2\beta^2 & \alpha\beta^3 & \beta^4 \\ 4\alpha^3\gamma & \alpha^2(\alpha\delta+3\beta\gamma) & \mp 2\alpha\beta(\alpha\delta+\beta\gamma) & \beta^2(3\alpha\delta+\beta\gamma) & 4\beta^3\delta \\ -6\alpha^2\gamma^2 + & -3\alpha\gamma(\alpha\delta+\beta\gamma) + & \alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2 =+ & -3\beta\delta(\alpha\delta+\beta\gamma) + & -6\beta^2\delta^2 + \\ 4\alpha\gamma^3 & \gamma^2(3\alpha\delta+\beta\gamma) & -2\delta\gamma(\alpha\delta+\beta\gamma) & \delta^2(\alpha\delta+3\beta\gamma) & 4\beta\delta^3 \\ \gamma^4 & \gamma^3\delta & -\delta^2\gamma^2 & \delta^3\gamma & \delta^4 \end{pmatrix}}_{A} \begin{pmatrix} F_{02200} \\ F_{02110} \\ F_{1110} \\ F_{20110} \\ F_{20020} \end{pmatrix}$$

$$G = A \cdot F$$

Passage:

$$P_1 := \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && -1 && \\ &&& 1 & \\ &&&& 1 \end{pmatrix}$$

$$P_2 := \begin{pmatrix} 1 &&&& \\ & 4 &&& \\ && 6 && \\ &&& 4 & \\ &&&& 1 \end{pmatrix} \qquad P := P_1P_2 = \begin{pmatrix} 1 &&&& \\ & 4 &&& \\ && -6 && \\ &&& 4 & \\ &&&& 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 &&&& \\ & 1/4 &&& \\ && -1/6 && \\ &&& 1/4 & \\ &&&& 1 \end{pmatrix}$$

−2 8 −1

$$A' := P^{-1}AP = \frac{1}{\chi(\alpha\delta-\beta\gamma)^{4\to 2}} \begin{pmatrix} \alpha^4 & 4\alpha^3\beta & 6\alpha^2\beta^2 & 4\beta^3\alpha & \beta^4 \\ \alpha^3\gamma & \alpha^2(\alpha\delta+3\beta\gamma) & 3\alpha\beta(\alpha\delta+\beta\gamma) & \beta^2(3\alpha\delta+\beta\gamma) & \delta\beta^3 \\ \alpha^2\gamma^2 & 2\alpha\gamma(\alpha\delta+\beta\gamma) & \alpha^2\delta^2+4\alpha\beta\delta\gamma+\beta^2\gamma^2 & 2\beta\delta(\alpha\delta+\beta\gamma) & \delta^2\beta^2 \\ \alpha\gamma^3 & \gamma^2(3\alpha\delta+\beta\gamma) & 3\delta\gamma(\alpha\delta+\beta\gamma) & \delta^2(\alpha\delta+3\beta\gamma) & \delta^3\beta \\ \gamma^4 & 4\delta\gamma^3 & 6\delta^2\gamma^2 & 4\delta^3\gamma & \delta^4 \end{pmatrix}$$

$\beta \leftrightarrow \gamma$

→ forme quartique $\beta \leftrightarrow \gamma$ □

$G' = A'F'$

$G' = P^{-1}AP\,F'$

$\underset{G}{PG'} = A\underset{F}{PF'}$

Six formes normales pour F':

	F'_{02200} a_0	F'_{02110} a_1	F'_{1110} a_2	F'_{20110} a_3	F'_{20020} a_4
I	1	0	$\mu \neq \pm 2$	0	1
II	0	0	1	0	1
III	0	0	1	0	0
IV	0	0	0	1	0
V	0	0	0	0	1
VI	0	0	0	0	0
mult	1	4	−6	4	1
	F_{02200}	F_{02110}	F_{1110}	F_{20110}	F_{20020}

OBS: Seul $F_{1110} \mapsto -F_{1110}$ et $G_{1110} \mapsto -G_{1110}$

Directement action sur la forme linéaire:

$G_{02200}X^4 + 1\,G_{02110}X^3Y \pm 1\,G_{1111}X^2Y^2 + 1\,G_{20110}XY^3 + 1\,G_{20020}Y^4$

Ordre 4 :

$$Z = c + xa + Yb + 0 + F_{20110}\,x^2ab + F_{20020}\,x^2b^2$$
$$+\ 0 + F_{11110}\,xYab + 0$$
$$+ F_{02200}\,Y^2a^2 + F_{02110}\,Y^2ab$$

(e_1: F_{02200}, e_2: F_{02110}, e_3: F_{11110}, e_4: F_{20110}, e_5: F_{20020})

$$z' = c' + x'a' + Y'b' + 0 + F'_{20110}\,x'^2a'b' + F'_{20020}\,x'^2b'^2$$
$$+\ 0 + F'_{11110}\,x'Y'a'b' + 0$$
$$+ F'_{02200}\,Y'^2a'^2 + F'_{02110}\,Y'^2a'b'$$

Symétrie: $x \leftrightarrow Y$, $a \leftrightarrow b$

g_{-2}	g_{-1}	g_0	g_1	g_2
1	4	5	4	1

Isotropie :

g_0: $\theta = 2$ — $f_{100} = \alpha$, $f_{010} = \beta$, $g_{100} = \gamma$, $g_{010} = \delta$, $h_{001} = \chi$

g_1: $\theta = 3$ — $f_{001} = \lambda$, $g_{001} = \mu$, $h_{101} = \varphi$, $h_{011} = \psi$

g_2: $\theta = 4$ — $h_{002} = \omega$

$$\begin{pmatrix} G_{20110} \\ G_{20020} \\ G_{11110} \\ G_{02200} \\ G_{02110} \end{pmatrix} \begin{matrix} e_4 \\ e_5 \\ e_3 \\ e_1 \\ e_2 \end{matrix} = \frac{1}{\chi(\alpha\delta-\beta\gamma)^2} \begin{pmatrix} \delta^2(\alpha\delta+3\beta\gamma) & 4\beta\delta^3 & -2\delta\gamma(\alpha\delta+\beta\gamma) & 4\alpha\gamma^3 & \gamma^2(3\alpha\delta+\beta\gamma) \\ \delta^3\gamma & \delta^4 & -\delta^2\gamma^2 & \gamma^4 & \delta\gamma^3 \\ -3\beta\delta(\alpha\delta+\beta\gamma) & -6\beta^2\delta^2 & \alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2 & -6\alpha^2\gamma^2 & -3\alpha\gamma(\alpha\delta+\beta\gamma) \\ \alpha\beta^3 & \beta^4 & -\alpha^2\beta^2 & \alpha^4 & \alpha^3\beta \\ \beta^2(3\alpha\delta+\beta\gamma) & 4\beta^3\delta & -2\alpha\beta(\alpha\delta+\beta\gamma) & 4\alpha^3\gamma & \alpha^2(\alpha\delta+3\beta\gamma) \end{pmatrix} \begin{pmatrix} F_{20110} \\ F_{20020} \\ F_{11110} \\ F_{02200} \\ F_{02110} \end{pmatrix}$$

$$\begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix} \rightsquigarrow I_5 + \varepsilon \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \rightsquigarrow I_5 + \varepsilon \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \rightsquigarrow I_5 + \varepsilon \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

Discrete Equivalences

Summarize ideas

$$0 + F_{20110}\, x^2 ab + F_{20020}\, x^2 b^2$$

$$+\ 0 + F_{11110}\, xyab + F_{11020}\, xyb^2 \qquad \curvearrowright \qquad \text{same with } F \mapsto G,\ \ x,y,a,b \mapsto x', y', a', b'$$

$$+ F_{02200}\, y^2 a^2 + F_{02110}\, y^2 ab + 0$$

Cas I:

$$1 \cdot y^2 a^2 + 0 + \underbrace{F_{11110}}_{M} xyab + 0 + 1\, x^2 b^2 \longrightarrow 1 \cdot y'^2 a'^2 + 0 + \underbrace{G_{11110}}_{M'} x'y'a'b' + 0 + 1\, x'^2 b'^2$$

$$\text{groupe} = \left\{ \left[\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \right], x \right\}$$

Équations à satisfaire: $\Delta := \alpha\delta - \beta\gamma$

$$0 \overset{02200}{=} -\frac{x}{\Delta^2}\left(G_{11110}\beta^2 x \delta^2 - \beta^4 x - x\delta^4 + \alpha^2\delta^2 - 2\alpha\beta\gamma\delta + \beta^2\gamma^2\right)$$

réordonner!

$$0 \overset{20110}{=} \frac{2}{\Delta^2} x^2 \left(G_{11110}\alpha^2\delta\gamma + G_{11110}\alpha\beta\gamma^2 - 2\alpha^3\beta - 2\delta\gamma^3\right) \xrightarrow{\text{solve}} \left\{ \beta := -\frac{\delta\gamma}{\alpha} \frac{(G_{11110}\alpha^2 - 2\gamma^2)}{(G_{11110}\gamma^2 - 2\alpha^2)} \right\}$$

$$0 \overset{11110}{=} -\frac{x}{\Delta^2}\left(-G_{11110}\alpha^2 x\delta^2 - 4G_{11110}\alpha\beta x\delta\gamma - G_{11110}\beta^2 x\gamma^2 + F_{11110}\alpha^2\delta^2 - 2F_{11110}\alpha\beta\delta\gamma + F_{11110}\beta^2\gamma^2 + 6\alpha^2\beta^2 x + 6x\delta^2\gamma^2\right)$$

$$0 \overset{20110}{\underset{02110}{=}} \frac{2}{\Delta^2} x^2 \left(G_{11110}\alpha\beta\delta^2 + G_{11110}\beta^2\delta\gamma - 2\alpha\beta^3 - 2\delta^3\gamma\right) \xrightarrow{\text{solve}} \gamma := -\frac{\alpha\beta}{\delta} \frac{(G_{11110}\delta^2 - 2\beta^2)}{(G_{11110}\beta^2 - 2\delta^2)} \longleftarrow \text{ICI}$$

- cas $\delta = 0$
- cas $\delta \neq 0$
- cas $G_{11110}\beta^2 - 2\delta^2 = 0$
- cas $G_{11110}\beta^2 - 2\delta^2 \neq 0$

$$0 \overset{20020}{=} -\frac{x}{\Delta^2}\left(G_{11110}\alpha^2 x\gamma^2 - x\alpha^4 - x\gamma^4 + \alpha^2\delta^2 - 2\alpha\beta\gamma\delta + \beta^2\gamma^2\right)$$

$\delta = 0$:

$$0 \overset{20200}{=} -\beta^2(\beta^2 x - \gamma^2)$$
$$0 \overset{20110}{=} \alpha\beta(G_{11110}\gamma^2 - 2\alpha^2)$$
$$0 \overset{11110}{=} \beta^2(-G_{11110}x\gamma^2 + F_{11110}\gamma^2 + 6x\alpha^2)$$
$$0 \overset{02110}{=} -2\alpha\beta^3$$
$$0 \overset{20020}{=} G_{11110}\alpha^2 x\gamma^2 - \alpha^4 x - x\gamma^4 + \beta^2\gamma^2$$

$\left(x, \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}\right)$ donc $\beta \neq 0$

donc $\alpha = 0$ $\quad \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}$

puis $x = \frac{\gamma^2}{\beta^2}$

So:

$$0 = 0$$
$$0 = 0$$
$$0 = \gamma^2(F_{11110}\beta^2 - G_{11110}\gamma^2)$$
$$0 = 0$$
$$0 = \frac{\gamma^2(\beta-\gamma)(\beta+\gamma)(\beta^2+\gamma^2)}{\beta^2}$$

donc: $\gamma = \beta$: $\begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}, 1$ $\quad$ $\gamma = -\beta$: $\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}, 1$ $\quad$ $\gamma = i\beta$: $\begin{bmatrix} 0 & \beta \\ i\beta & 0 \end{bmatrix}, -1$ $\quad$ $\gamma = i\beta$: $\begin{bmatrix} 0 & \beta \\ -i\beta & 0 \end{bmatrix}, -1$

et alors $G_{11110} = 1\ F_{11110}$
1
-1
-1

-50-

$G_{11110}\beta^2 - 2\delta^2 = 0$; sous cas $G_{11110} = 0$ force $\delta = 0$ déjà traité donc ops $G_{11110} \neq 0$

supposer $\delta = \frac{1}{2}\sqrt{2}\sqrt{G_{11110}}\,\beta$ Rence $\beta \neq 0$ car $\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix} \in GL_2$

$0 \overset{02110}{=} \frac{1}{2}\alpha\beta^3(G_{11110} - 2)(G_{11110} + 2)$ force $\alpha = 0$

les éq^s deviennent :

$0 \overset{20200}{=} \frac{\beta^2}{4}\left(G_{11110}^2\beta^2 x - 4\beta^2 x + 4\gamma^2\right)$

$0 \overset{20110}{=} -\sqrt{2}\sqrt{G_{11110}}\,\beta\gamma^3$ ⟶ force $\gamma = 0$

$\begin{bmatrix} 0 & \beta \\ 0 & \frac{1}{2}\sqrt{2}\sqrt{G_{11110}}\beta \end{bmatrix}$

$0 \overset{11110}{=} \beta^2\gamma^2(2G_{11110}x + F_{11110})$

$0 \overset{02110}{=} 0$

$0 \overset{20020}{=} \gamma^2(-x\gamma^2 + \beta^4)$

Ensuite: $0 =$

$0 =$

$0 =$ Stop ⟶ contradict $\in GL_2$

$0 = 0$

$0 =$

CCL : Ce cas $G_{11110}\beta^2 - 2\delta^2 = 0$ est impossible □

-50-1/2-

Cas $\delta \neq 0 \neq G_{11110}\beta^2 - 2\delta^2$: on résout donc $\gamma := -\frac{\alpha\beta}{\delta}\frac{(G_{11110}\delta^2 - 2\beta^2)}{(G_{11110}\beta^2 - 2\delta^2)}$ alors $\Delta = \frac{2\alpha}{\delta}\frac{(G_{11110}\beta^2\delta^2 - \beta^4 - \delta^4)}{(G_{11110}\beta^2 - 2\delta^2)}$ must be $\neq 0$ → must be $\neq 0$; $\delta \neq 0$; $\neq 0$

on obtient:

$$0 \overset{20110}{=} -\frac{4\alpha^3\beta(\beta-\delta)(\beta+\delta)(\beta^2+\delta^2)(G_{11110}-2)(G_{11110}+2)(G_{11110}\beta^2\delta^2-\beta^4-\delta^4)}{\delta^2(G_{11110}\beta^2-2\delta^2)^3}$$

($G_{11110} \neq 0$, $G_{11110} \neq 0$, $G_{11110}\beta^2\delta^2-\beta^4-\delta^4 \neq 0$, $\delta^2 \neq 0$, $G_{11110}\beta^2-2\delta^2 \neq 0$)

donc:

$$0 = \beta(\beta-\delta)(\beta+\delta)(\beta^2+\delta^2)$$

cas $\beta = 0$: d'où $\gamma = 0$:

$0 \overset{20200}{=} \delta^2(-x\delta^2+\alpha^2)$ ⟶ solve $x := \frac{\alpha^2}{\delta^2}$ d'où: $0 = 0$

$0 \overset{20110}{=} 0$ — $0 = 0$

$0 \overset{11110}{=} \alpha^2\delta^2(-G_{11110}x + F_{11110})$ — $0 = \alpha^2(F_{11110}\delta^2 - G_{11110}\alpha^2)$

$0 \overset{02110}{=} 0$ — $0 = 0$

$0 \overset{20020}{=} -\alpha^2(\alpha^2 x - \delta^2)$ — $0 = -\frac{\alpha^2}{\delta^2}(\alpha-\delta)(\alpha+\delta)(\alpha^2+\delta^2)$

$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$ ($\alpha \neq 0$, $\delta \neq 0$)

$\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}, \frac{\alpha^2}{\delta^2} \right\}$ ($\alpha \neq 0$, $\delta \neq 0$)

d'où $\delta = \begin{smallmatrix} 1 \\ -1 \\ i \\ -i \end{smallmatrix}\, \alpha$ et enfin $G_{11110} = \begin{smallmatrix} 1 \\ 1 \\ -1 \\ -1 \end{smallmatrix}\, F_{11110}$ DÉJÀ vu

Cas $\delta = \beta$: d'où $\gamma = -\alpha$:

$0 = \beta^2(G_{11110}x\beta^2 - 2\beta^2 x + 4\alpha^2)$ ($\beta \neq 0$) ⟶ solve $x \overset{20200}{=} -\frac{4\alpha^2}{\beta^2(G_{11110}-2)}$

puis:

$0 = 0$ — $0 = 0$

$0 = 2\alpha^2\beta^2(G_{11110}x + 2F_{11110} + 6x)$ — $\begin{bmatrix} \alpha & \beta \\ -\alpha & \beta \end{bmatrix}$ $\Delta = 2\alpha\beta$ ($\neq 0$, $\neq 0$) — $0 = 0$

$0 \overset{11110}{=} 4\alpha^2\left(\frac{F_{11110}G_{11110}\beta^2 - 2\beta^2F_{11110} - 2G_{11110}\alpha^2 - 12\alpha^2}{G_{11110}-2}\right)$

$0 = 0$ ($\neq 0$, $\neq 0$) — $0 = 0$

$0 = \alpha^2(G_{11110}\alpha^2 x - 2\alpha^2 x + 4\beta^2)$ ($\alpha \neq 0$) — $0 = -\frac{4\alpha^2}{\beta^2}(\alpha-\beta)(\alpha+\beta)(\alpha^2+\beta^2)$

$\beta := \alpha$ $\quad G_{11110} = 2\dfrac{(6+F_{11110})}{F_{11110}-2}$

$\beta := -\alpha$ $\quad$ pareil ↑

$\beta := I\alpha$ $\quad G_{11110} = 2\dfrac{(F_{11110}-6)}{F_{11110}+2}$

$\beta := -I\alpha$ $\quad$ pareil ↑

<u>Cas $\delta = -\beta$:</u> $\quad X \overset{2020}{=} \dfrac{-4\alpha^2}{\beta^2(G_{11110}-2)}$

puis $0 \overset{2020}{=} (\alpha-\beta)(\alpha+\beta)(\alpha^2+\beta^2) = 0$ $\quad$ puis $G =$ déjà vu

puis $0 \overset{11110}{=} 4\alpha^2 \dfrac{(F_{11110}G_{11110}\beta^2 - 2F_{11110}\beta^2 - 2G_{11110}\alpha^2 - 12\alpha^2)}{G_{11110}-2}$ déjà vu

différent

<u>Cas $\delta = I\beta$:</u> $\quad X \overset{2020}{=} -\dfrac{4\alpha^2}{\beta^2(G_{11110}+2)}$ $\quad$ puis $0 \overset{11110}{=} \dfrac{-4\alpha^2}{G_{11110}+2}\,\dfrac{(F_{11110}G_{11110}\beta^2 + 2F_{11110}\beta^2 - 2G_{11110}\alpha^2 + 12\alpha^2)}{G_{11110}+2}$

$0 \overset{2020}{=} (\alpha-\beta)(\alpha+\beta)(\alpha^2+\beta^2) \cdot \ast$

$\beta := \alpha$ $\quad G_{11110} = -2\dfrac{(F_{11110}+6)}{F_{11110}-2}$

$\beta := -\alpha$ $\quad G_{11110} =$ pareil ↑

$\beta := I\alpha$ $\quad G_{11110} = -2\dfrac{(F_{11110}-6)}{F_{11110}+2}$

$\beta := -I\alpha$ $\quad G_{11110} =$ pareil ↑

<u>Enfin, Cas $\delta = -I\beta$:</u> $\quad X \overset{2020}{=} -\dfrac{4\alpha^2}{\beta^2(G_{11110}+2)}$ $\quad$ puis $0 \overset{11110}{=} \dfrac{-4}{\alpha^2}\dfrac{(F_{11110}G_{11110}\beta^2 + 2\beta^2F_{11110} - 2\alpha^2G_{11110} + 12\alpha^2)}{G_{11110}+2}$ pareil

$0 \overset{2020}{=} \dfrac{4\alpha^2}{\beta^2}(\alpha-\beta)(\alpha+\beta)(\alpha^2+\beta^2)$

donc pareil que le cas $\delta = I\beta$ □

<u>CCL</u> :

- □ $G_{11110} = \pm F_{11110}$
- □ $G_{11110} = 2\dfrac{(6+F_{11110})}{F_{11110}-2}$ $\qquad$ □ $G_{11110} = -2\dfrac{(F_{11110}+6)}{F_{11110}-2}$ (-1)
- □ $G_{11110} = 2\dfrac{(F_{11110}-6)}{F_{11110}+2}$ $\qquad$ □ $G_{11110} = -2\dfrac{(F_{11110}-6)}{F_{11110}+2}$ (-1)

$z = c + xa + yb, \; i\,g^{r}al$

$0 \overset{20200}{=} -F_{20200} + G_{20200} + f_{101} - h_{002}$

$0 \overset{02200}{=} -F_{02200} + G_{02200}$

$0 \overset{20110}{=} -F_{20110} + G_{20110}$

$0 \overset{11110}{=} -F_{11110} + G_{11110} + f_{101} + g_{011} - 2h_{002}$

$0 \overset{02110}{=} -F_{02110} + G_{02110}$

$0 \overset{20020}{=} -F_{20020} + G_{20020}$

$0 \overset{02020}{=} -F_{02020} + G_{02020} + g_{011} - h_{002}$

F_{20200}	F_{20110}	F_{20020}
F_{11200}	F_{11110}	F_{11020}
F_{02200}	F_{02110}	F_{02020}

Choix :

F_{20200}	F_{20110}	F_{20020}
F_{11200}	F_{11110}	F_{11020}
F_{02200}	F_{02110}	F_{02020}

Avant :

$f_{011} := 0$

$F_{11200} := 0 \qquad G_{11200} := 0$

$g_{101} := 0$

$F_{11020} := 0 \qquad G_{11020} := 0$

Solve ordre 4 :

$Eq4[11200] \leftarrow f_{011} \qquad X_{011}$

$Eq4[11020] \leftarrow g_{101} \qquad Y_{101}$

$Eq4[20\;20\;0] \leftarrow f_{101} \qquad X_{101}$

$Eq4[02\;02\;0] \leftarrow g_{011} \qquad Y_{011}$

5D-2PDE: $\mu \geq 5$:

$\acute{E}q[i,j,0,0,m]$ $i+j\geq 2$ $F_{i,j,0,0,m} := 0 \leftarrow$ $\boxed{R_{i,j,m}}$ $i+j+2m=\mu$ Restent $\{R_{00m}$ $2m=\mu=\Theta=2\nu$; $\{R_{10m}, R_{01m}$ $1+2m=\mu=\Theta=2\nu+1$

$\acute{E}q[i,j,1,0,m]$ $i+j\geq 2$ $F_{i,j,1,0,m} := 0 \leftarrow$ $\boxed{f_{i,j,m}}$ $i+j+2m=\mu-1$ Restent $\{f_{10m}, f_{01m}$ $1+2m=\mu-1=2\nu-1$; $\{f_{00m}$ $2m=\mu-1=2\nu$

$\acute{E}q[i,j,0,1,m]$ $i+j\geq 2$ $F_{i,j,0,1,m} := 0 \leftarrow$ $\boxed{g_{i,j,m}}$ $i+j+2m=\mu-1$ Restent $\{g_{10m}, g_{01m}$ $1+2m=\mu-1=2\nu-1$; $\{g_{00m}$ $1+2m=\mu-1=2\nu$

$\mu=5$: $f_{ij0} := 0$ $i+j=4$

$g_{ij0} := 0$ $i+j=4$

Restent: R_{102} R_{012}

f_{002}

g_{002}

$f_{201} := 2R_{102}$ $g_{201} := 0$

$f_{111} := 2R_{012}$ $g_{111} := 2R_{102}$

$f_{021} := 0$ $g_{021} := 2R_{012}$

30200	$0 = -F_{30200} + G_{30200} + R_{102}$
21200	$0 = -F_{21200} + G_{21200} + R_{012}$
• $\boxed{20300}$	$0 = -F_{20300} + G_{20300} + f_{002}$
$=\boxed{21110}$	$0 = -F_{21110} + G_{21110} + 2R_{102}$
$=\boxed{12110}$	$0 = -F_{12110} + G_{12110} + 2R_{012}$
20210	$0 = -F_{20210} + G_{20210} + g_{002}$
– 11210	$0 = -F_{11210} + G_{11210} + 2f_{002}$
12020	$0 = -F_{12020} + G_{12020} + R_{102}$
# 03020	$0 = -F_{03020} + G_{03020} + R_{012}$
– 11120	$0 = -F_{11120} + G_{11120} + 2g_{102}$
02120	$0 = -F_{02120} + G_{02120} + f_{002}$
• $\boxed{02030}$	$0 = -F_{02030} + G_{02030} + g_{002}$

$\mu = 8 = 2\cdot 4$:

$0 \overset{20202}{=} -F + G + 3\,f_{103} - 6\,h_{004}$

$0 \overset{11202}{=} -F + G + 3\,f_{013}$

$0 \overset{11022}{=} -F + G + 3\,g_{103}$

$0 \overset{02022}{=} -F + G + 3\,g_{013} - 6\,h_{004}$

add

Sélectionner:

$0 \overset{30301}{=} -F + G + 3\,f_{103} - 4\,h_{004}$

$\left[\,0 \overset{03031}{=} -F + G + 3\,g_{013} - 4\,h_{004}\,\right]$

Impossible de sélectionner:

$0 \overset{20112}{=} -F + G + 3\,g_{103}$

$0 \overset{20022}{=} -F + G$

$0 \overset{11112}{=} -F + G + 3\,f_{103} + 3\,g_{013} - 12\,h_{004}$ IMPOSSIBLE

$0 \overset{02202}{=} -F + G$

$0 \overset{02112}{=} -F + G + 3\,f_{013}$

CCL:

$\mu = 2\nu$ pair:		$\mu = 2\nu+1$ impair:	
$2020\,\nu{-}2$	$f_{10\nu-1}$	$2030\,\nu{-}2$	$f_{00\nu}$
$1120\,\nu{-}2$	$f_{01\nu-1}$	$0203\,\nu{-}2$	$g_{00\nu}$
$1102\,\nu{-}2$	$g_{10\nu-1}$	$3020\,\nu{-}2$	$h_{10\nu}$
$0202\,\nu{-}2$	$g_{01\nu-1}$	$0302\,\nu{-}2$	$h_{01\nu}$
$3030\,\nu{-}3$	$h_{00\nu}$		

Isote

ω h_{002} Z_{002}

σ g_{001} X_{001}

τ y_{001} Y_{001}

ϕ h_{101} Z_{101}

ψ h_{011} Z_{011}

α g_{100} X_{100} β g_{010} X_{010}

γ y_{100} Y_{100} δ y_{010} Y_{010}

χ h_{001} Z_{001}

Iso-5D-2PDE

dim	1	4	5	4	1
	$\mathfrak{g}_{-2}$ ⊕	$\mathfrak{g}_{-1}$ ⊕	$\mathfrak{g}_0$ ⊕	$\mathfrak{g}_1$ ⊕	$\mathfrak{g}_2$
ordre	0	1	2	3	4

<u>$\mathfrak{h}\mathfrak{g}$:</u>

ordre 2 | 3 | 4

$$f \equiv \alpha x + \beta y \;\Big|\; + \lambda z$$

$$g \equiv \gamma x + \delta y \;\Big|\; + \mu z$$

$$h \equiv xz \;\Big|\; + \phi xz + \psi yz \;\Big|\; + \omega z^2$$

<u>$V\mathfrak{g}$:</u>

ordre 2 | 3 | 4

$$XX \equiv \boxed{X_{100}} \cdot x + \boxed{X_{010}} \cdot y \;\Big|\; + \boxed{X_{001}} \cdot z$$

$$YY \equiv \boxed{Y_{100}} \cdot x + \boxed{Y_{010}} \cdot y \;\Big|\; + \boxed{Y_{001}} \cdot z$$

$$ZZ \equiv \boxed{Z_{001}} \cdot z \;\Big|\; + \boxed{Z_{101}} \cdot xz + \boxed{Z_{011}} \cdot yz \;\Big|\; + \boxed{Z_{002}}\, z^2$$