Moving frames and invariants for curves in parabolic homogeneous spaces
Lecture 2

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Outline

1. **Revised definitions**
   - Model curves
   - Projective parametrization
2. **Construction of the canonical moving frame**
   - Normalization conditions
   - Existence of the normal moving frame
3. **Examples and discussion**
   - Classical examples
   - Ruled surfaces
   - G_2 examples
   - Generalizations

Symbol of curves in parabolic homogeneous spaces

- Let $M = G/P$ be an arbitrary parabolic homogeneous space: $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a graded semisimple Lie algebra of the Lie group G and $\mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$.
- M is naturally equipped with a a structure of a filtered manifold

$$0 \subset T^{-1}M \subset \cdots \subset T^{-\nu}M = TM$$

  defined as a flag of G-invariant vector distributions equal to $\oplus_{i\leq k}\mathfrak{g}_{-i}$ mod $\mathfrak{p}$ at $o = eP$.
- Given a curve $\gamma \subset M$ we define its symbol $\mathfrak{n}$ at $x \in \gamma$ as $\operatorname{gr} T_x\gamma$, which is a 1-dimensional graded subspace in $\mathfrak{g}_{-k}$ for some $k > 0$. Here k is a minimal positive integer such that γ is an integral curve of $T^{-k}M$.
- As the symbol is well-defined modulo the action of G_0 on $\mathfrak{g}_-$, we can say that the symbol of a curve is just a homogeneos element $X \in \mathfrak{g}_-$ viewed up to the action of G_0 and the non-zero scale. Here $\mathfrak{n} = \langle X \rangle$ is the corresponding 1-dimensional graded subalgebra in $\mathfrak{g}_-$.

- The group G_0 acts on $P(\mathfrak{g}_{-k})$ (for any k) with a finite number of orbits [E.Vinberg, *The Weyl group of a graded Lie algebra*, English transl. Math. USSR, Izvestija **10**(1976)]. So, possibly restricting to an open subset on a curve, we can always assume that the symbol of a curve is constant. *NB. This is different for submanifolds of higher dimension, where the assumption of constant symbol is required.*
- (Intrinsic) prolongation of $\mathfrak{n} = \langle X \rangle$ in $\mathfrak{g}$ is a largest graded subalgebra $\mathrm{Prol}(\mathfrak{n})$ of $\mathfrak{g}$ such that $\mathrm{Prol}_{-}(\mathfrak{n}) = \mathfrak{n}$. It can be constructed inductively as:

$$\mathrm{Prol}_i(\mathfrak{n}) = \mathfrak{n}_i, \quad (i < 0),$$
$$\mathrm{Prol}_i(\mathfrak{n}) = \{u \in \mathfrak{g}_i \mid [X, u] \subset \mathrm{Prol}_{i-1}(\mathfrak{n})\}, \quad (i \geq 0).$$

- **Theorem**. *We have* $\dim \mathrm{sym}(N) \leq \dim \mathrm{Prol}(\mathfrak{n})$*. Moreover the equality is achieved if and only if N is locally equivalent to the orbit of the subgroup* $\exp(tX) \subset G$ *through $o = eP$.*

- Element $X \in \mathfrak{g}_{-k}$ can always be completed by elements $H \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_k$ to the basis of $\mathfrak{sl}_2$ subalgebra (Morozov, Vinberg). Then by definition we have $\langle X, H, Y \rangle \subset \mathrm{Prol}(\langle X \rangle)$.
- The prolongation $\mathrm{Prol}(\langle X \rangle)$ can be described as a the semidirect sum of $\mathfrak{sl}_2$ and all $\mathfrak{sl}_2$ sumbmodules in $\mathfrak{g}$ concentrated in non-negative degree.
- The subgroup corresponding to $\langle X, H, Y \rangle$ acts transitively on the closure of the orbit $\exp(tX).o$. Thus, any flat curve is always rational curve or its cover.

Notion of a normal moving frame

- $\gamma \subset G/P$ is any curve with the symbol $X \in \mathfrak{g}_{-k}$
- $\pi\colon G \to G/P$ is the standard principle P-bundle, and $Q_{-1} = \pi^{-1}(\gamma)$
- $\omega\colon TG \to \mathfrak{g}$ is the left-invariant Maurer-Cartan form on G
- $\pi|_{Q_{-1}}\colon Q_{-1} \to \gamma$ is the restriction of the principle P-bundle to γ
- *Moving frame* is (any) subbundle of this bundle: $E \subset Q_{-1}$
- We construct a *normal moving frame* by imposing conditions on $\omega(T_z E) \subset \mathfrak{g}$ for $z \in E$
- Example (Cartan) of such conditions for projective curves in P^2:

$$\omega|_E = \begin{pmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & 0 & \omega_{01} \\ 0 & \omega_{10} & -\omega_{00} \end{pmatrix}$$

Normalization conditions

- Define normalization conditions for any curve of symbol $X \in \mathfrak{g}_{-k}$ as a graded subspace $W \subset \mathfrak{g}$ such that:

$$
\begin{aligned}
&(1) \quad W_i = 0 \text{ for } i < 0; \\
&(2) \quad W_i \text{ is complementary to } \operatorname{Prol}_i(X) + [X, \mathfrak{g}_{i+k}]; \\
&(3) \quad W \text{ is invariant with respect to } \operatorname{Prol}^{(0)}(X) = \sum_{i \geq 0} \operatorname{Prol}_i(X).
\end{aligned}
$$

- Such W may or may not exist. It **always exists if** *Prol*(X) **is reductive**:

$$W = \{u \in \operatorname{Prol}(X)^{\perp}_{\geq 0} \mid [u, Y] = 0\},$$

where $\operatorname{Prol}(X)^{\perp}$ is the orthogonal complement to $\operatorname{Prol}(X)$ w.r.t. to Killing form of $\mathfrak{g}$, and $\operatorname{Prol}(X)^{\perp}_{\geq 0}$ is its part of non-negative degree.

Main result

Theorem

Fix normalization conditions for a given integral curve of symbol $X \in \mathfrak{g}_{-k}$. Then there exists a unique moving frame $E \to \gamma$:

$$\omega(T_z E) \subset \mathrm{Prol}(X) \oplus W \quad \text{for all } z \in E.$$

Decompose $\omega|_E$ as $\omega_I + \omega_{II}$ according to the decomposition $\mathrm{Prol}(X) \oplus W$.

- *The form ω_I is a flat projective connection on γ. It defines the canonical projective parameter on γ.*
- *The form ω_{II} is a vertical equivariant form defining the complete system of fundamental invariants of γ. In particular, γ is locally flat if and only if $\omega_{II} = 0$.*

Ideals of the proof

- B.D, I. Zelenko, *Geometry of curves in generalized flag varieties*, Transformation groups, (6) 2013, `arXiv:1110.0226`.
- The idea is to start from the principal P-bundle $\pi\colon Q_{-1} \to \gamma$, where $Q_{-1} = \pi^{-1}(\gamma)$ and reduce it to a series of principal subbundles $\pi_k\colon Q_k \to \gamma$, each with its own structure group having the Lie algebra

$$\sum_{i=0}^{k} \mathrm{Prol}_i(X) + \sum_{j>k} \mathfrak{g}_j.$$

- At each step $k \geq 0$ we define Q_k as the set of all such points $z \in Q_{k-1}$, that

$$\mathrm{Im}\,\omega_z \subset \mathrm{Prol}(X) \oplus W \mod \sum_{i \geq k} \mathfrak{g}_i.$$

- Non-degenerate curves in P^n can be naturally lifted to integral curves in the flag variety $F_{1,2,\dots,n}(\mathbb{R}^{n+1})$. The corresponding curve type is given by:

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

  Distinguished curves are osculating flags of rational normal curves. The prolongation is $\mathrm{Prol}(X) = \langle X, H, Y \rangle = \mathfrak{sl}_2$ — the irreducible embedding of $\mathfrak{sl}_2$. Normalization condition $W = \langle Y^2, \dots, Y^n \rangle$.
- Any curves in conformal spaces $S^n = SO(n+1,1)/P$ approximated by conformal circles.
- Curves of generic type in $\mathrm{Gr}_n(\mathbb{R}^{2n})$, $\mathrm{IGr}_n(\mathbb{R}^{2n})$ both for $Sp(2n,\mathbb{R})$ and $SO(n,n)$.
- Curves of generic type in $\mathrm{Gr}_n(\mathbb{R}^{rn})$ for any $r \geq 2$. They correspond to linear systems of r ODEs of order n. (Se-ashi)

No invariant normalization conditions

- What if invariant normalization conditions do not exist?
- Take any graded subspace W complementary to $\mathrm{Prol}(X) + [X, \mathfrak{g}]$. We can still define *the normal moving frame bundle*, but it is no longer a principal fiber bundle. Thus, Cartan connection does not survive, and projective parametrization is not defined uniformly for all curves of symbol X.
- Smallest example without invariant space W is curves in $\mathrm{Gr}_2(\mathbb{R}^5) =$ ruled surfaces in P^4.
- More generally, the normalization condition does exist for ruled surfaces in P^{2k+1} and fails to exist for ruled surfaces in P^{2k} for generic curve type.

G_2 examples

- Take contact parabolic geometry of $G = G_2$.
- $\mathfrak{g}_{-1} \equiv S^3(\mathbb{R}^2)$ under the irreducible action of $G_0 \equiv GL(2, \mathbb{R})$;
- One generic (non-contact) curve type and three types of contact curves depending on the multiplicity of roots of cubic polynomial. Normalization conditions always exist:
- **Non-contact curves:** symmetry algebra $\mathrm{Prol}(X) = \mathfrak{sl}_2 + \mathfrak{sl}_2$ and a cubic binary form as as fundamental invariant.
- **Triple root:** curve is naturally lifted to G/B; after the lift (in the non-degenerate case) we get the symmetry algebra $\mathrm{Prol}(X) = \mathfrak{sl}_2$ and 1 fundamental invariant.
- **Double root:** symmetry algebra $\mathrm{Prol}(X)$ is 5-dimensional and is non-reductive. Yet, normalization conditions do exist, and there are 2 fundamental invariants.
- **No multiple roots:** symmetry algebra $\mathrm{Prol}(X) = \mathfrak{sl}_2$, there are 3 fundamental invariants.

Generalizations

- **Curves in (curved) parabolic geometries.** Main result stays the same, as moving frames for curves in Cartan geometries work as in homogeneous case.
- **Submanifolds of higher dimension in parabolic homogeneous spaces.** Needs modification of normalization conditions: W is an $\operatorname{Prol}(X)^{(0)}$-invariant splitting of two exact sequences:

$$0 \to \operatorname{Prol}(X) \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Prol}(X) \to 0.$$

$$0 \to B^1_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) \to Z^1_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) \to H^1_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) \to 0.$$