GRIEG LECTURE 1

JOËL MERKER

www.imo.universite-paris-saclay.fr/∼merker/

Département de Mathématique d'Orsay

Université Paris-Saclay

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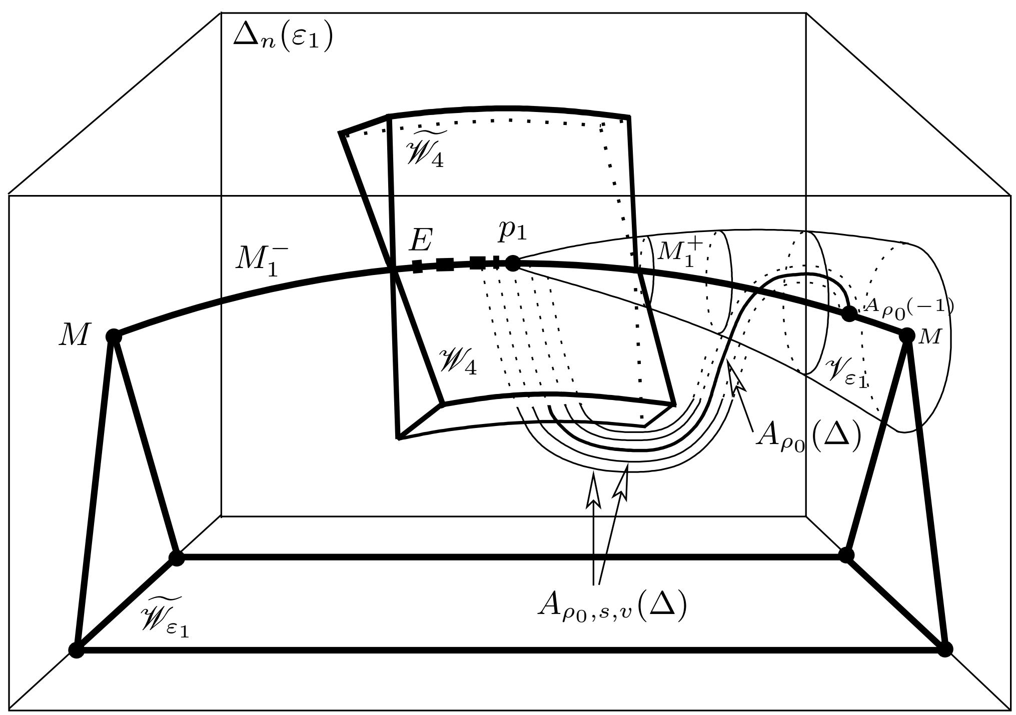


Model Quadric $M^3 \subset \mathbb{C}^2$

- **Complex space :**

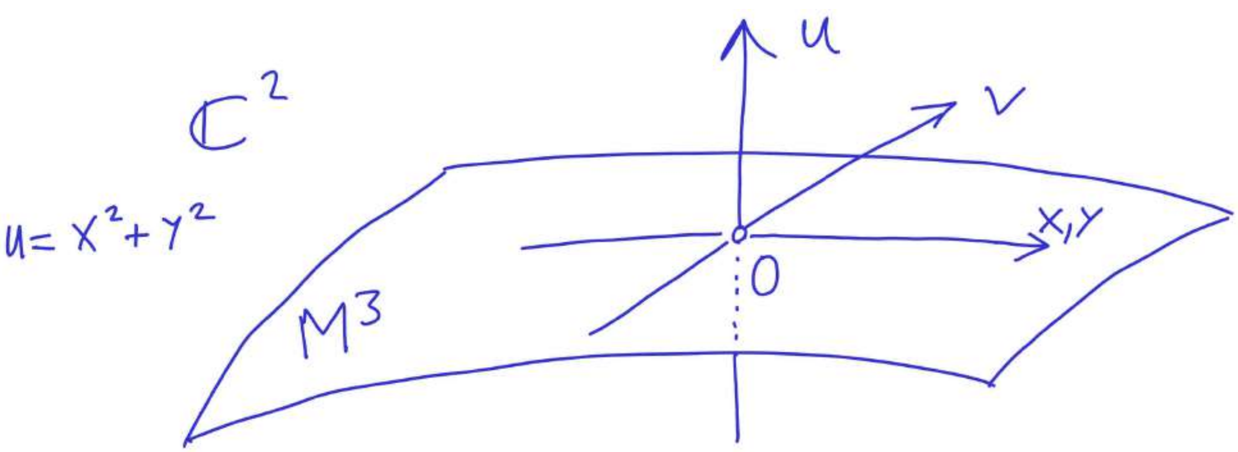
$$\mathbb{C}^2 \ni (z, w) = \big(x + \sqrt{-1}\, y,\; u + \sqrt{-1}\, v\big).$$

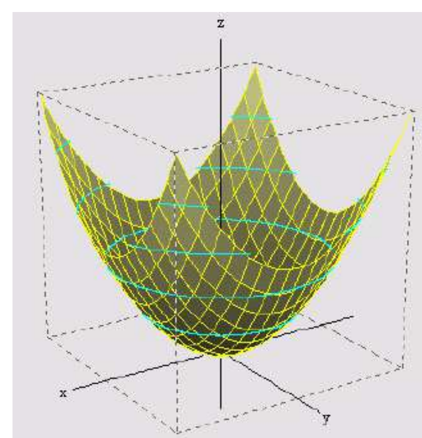
- **Model quadric hypersurface :**

$$\mathsf{Re}\, w = z\,\overline{z},$$

that is :

$$u = x^2 + y^2.$$





- **Affine counterpart :** Surface :

$$S^2 := \Big\{(x,y,u) \in \mathbb{R}^3 : \ \ u = x^2 + y^2\Big\}.$$

- **Affine group :**

$$\begin{aligned} x' &= \mathsf{a}\,x + \mathsf{b}\,y + \mathsf{c}\,u + \mathsf{d}, \\ y' &= \mathsf{k}\,x + \mathsf{l}\,y + \mathsf{m}\,u + \mathsf{n}, \\ u' &= \mathsf{p}\,x + \mathsf{q}\,y + \mathsf{r}\,u + \mathsf{s}, \end{aligned} \qquad 0 \neq \begin{vmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ \mathsf{k} & \mathsf{l} & \mathsf{m} \\ \mathsf{p} & \mathsf{q} & \mathsf{r} \end{vmatrix}.$$

- **Doubrov-Komrakov-Rabinovich 1996 & Eastwood-Ezhov 1999 :**

Theorem 1. *Each locally homogeneous surface in the three-dimensional affine geometry is an open subset of a quadric or cylinder, or is equivalent to an open subset of one of the following surfaces:*

(1) $z = x^\alpha y^\beta$.
(2) $z = (x^2+y^2)^\beta e^{\alpha \arg(ix+y)}$.
(3) $z = \ln x + \alpha \ln y$.

(4) $z = \alpha \arg(ix + y) + \beta \ln(x^2 + y^2)$, $\alpha = 1$, $\beta \geq 0$ *or* $\alpha = 0$, $\beta = 1$.
(5) $z = x(\alpha \ln x + \ln y)$.
(6) $(z - xy + x^3/3)^2 = \alpha(y - x^2/2)^3$, $\alpha \neq 0$.
(7) $z = y^2 \pm e^x$.
(8) $z = y^2 \pm x^\alpha$.
(9) $z = y^2 \pm \ln x$.
(10) $z = y^2 \pm x \ln x$.
(11) $z = xy + e^x$.
(12) $z = xy + x^\alpha$.
(13) $z = xy + \ln x$.
(14) $z = xy + x \ln x$.
(15) $z = xy + x^2 \ln x$.
(16) $xz = y^2 \pm z^\alpha$.
(17) $xz = y^2 \pm x \ln x$.
(18) $xz = y^2 \pm x^2 \ln x$.

- **Open classification problems :**
  - ☐ $S^2 \subset \mathbb{R}^4$ [Vitushkin's school $\longmapsto M^6 \subset \mathbb{C}^4$ **Hard !**]
  - ☐ $S^3 \subset \mathbb{R}^4$ *nondegenerate* [Wermann, Eastwood-Ezhov]
  - ☐ $S^3 \subset \mathbb{R}^4$ *degenerate :* Hessian rank 2, rank 1.

- **Infinitesimal affine symmetries :**

$$\begin{aligned} L \;:=\; & \big(\mathsf{A}\,x + \mathsf{B}\,y + \mathsf{C}\,u + \mathsf{D}\big)\,\partial_x \\ & + \big(\mathsf{K}\,x + \mathsf{L}\,y + \mathsf{M}\,u + \mathsf{N}\big)\,\partial_y \\ & + \big(\mathsf{P}\,x + \mathsf{Q}\,y + \mathsf{R}\,u + \mathsf{S}\big)\,\partial_u. \end{aligned}$$

- **Observation.** [Lie]

flow $\exp(t\,L)(x,y,u)$ stabilizes S^2 $\quad\Longleftrightarrow\quad$ $L\big|_{S^2}$ tangent to S^2.

- **Compute :**

$$\mathsf{eqL} \;:=\; L\,\big(-u + x^2 + y^2\big)\Big|_{u=x^2+y^2}.$$

- **Weights :**

$$\begin{aligned} [x] &:= 1, \\ [y] &:= 1, \\ [u] &:= 2. \end{aligned}$$

- **Expand :**

$$\mathsf{eqL} \;=\; \sum_{\mu=0}^{\infty} \; \sum_{i+j=\mu} \mathsf{Coefficient}_{i,j}\, x^i\, y^j.$$

- **Increasing weights :**

$$\mu \;=\; 0,\;\; 1,\;\; 2,\;\; 3,\;\; 4,\;\; \ldots .$$

- **Specifically :**

$$\begin{aligned} 0 \;\equiv\; & -\mathsf{S} \\ & + \big(2\,\mathsf{D} - \mathsf{P}\big)\,x + \big(2\,\mathsf{N} - \mathsf{Q}\big)\,y \\ & + \big(2\,\mathsf{A} - \mathsf{R}\big)\,x^2 + \big(2\,\mathsf{B} + 2\,\mathsf{K}\big)\,xy + \big(2\,\mathsf{L} - \mathsf{R}\big)\,y^2 \\ & + \big(2\,\mathsf{C}\big)\,x^3 + \big(2\,\mathsf{C}\big)\,xy^2 + \big(2\,\mathsf{M}\big)\,x^2y + \big(2\,\mathsf{M}\big)\,y^3 \\ & + \mathbf{0} \text{ at higher orders.} \end{aligned}$$

- **Luck here :**

Stop at weight 4 !

- **Solve :**

$$
\begin{aligned}
\mathsf{S} &:= 0, \\
\mathsf{P} &:= 2\,\mathsf{D}, & \mathsf{Q} &:= 2\,\mathsf{N}, \\
\mathsf{R} &:= 2\,\mathsf{A}, & \mathsf{K} &:= -\,\mathsf{B}, & \mathsf{L} &:= \mathsf{A}, \\
\mathsf{C} &:= 0, & \mathsf{M} &:= 0.
\end{aligned}
$$

- **4 free constants :**

$$
\begin{aligned}
L = \quad & \big(\mathsf{A}\,x + \mathsf{B}\,y + \mathsf{D}\big)\,\partial_x + \big(-\,\mathsf{B}\,x + \mathsf{A}\,y + \mathsf{N}\big)\,\partial_y \\
& + \big(2\,\mathsf{D}\,x + 2\,\mathsf{N}\,y + 2\,\mathsf{A}\,u\big)\,\partial_u.
\end{aligned}
$$

- **4 generators :**

$$
\begin{aligned}
e_1 &:= x\,\partial_x + y\,\partial_y, \\
e_2 &:= y\,\partial_x - x\,\partial_y, \\
e_3 &:= \partial_x + 2\,x\,\partial_u, \\
e_4 &:= \partial_y + 2\,y\,\partial_u.
\end{aligned}
$$

- **Lie structure :**

$$
\big[e_1,\,e_3\big] = -\,e_3, \qquad \big[e_1,\,e_4\big] = -\,e_4, \qquad \big[e_2,\,e_3\big] = e_4, \qquad \big[e_2,\,e_4\big] = -\,e_3.
$$

A result

- **Hypersurface $H^3 \subset \mathbb{C}^4$:**

$$u = F(x, y, z).$$

- **Hessian :** [Its rank is affinely invariant]

$$\boldsymbol{H}_F := \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}.$$

- **Current hypothesis :**

$$1 = \mathsf{rank}\, \boldsymbol{H}_F.$$

- **Exclude :** [Homogeneity hypothesis]

$$u = \frac{x^2}{2} + \mathbf{0} + \mathrm{O}_{x,y,z}(4) \qquad \Longleftrightarrow \qquad H^3 \cong C^1 \times \mathbb{C}^2,$$

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + \mathbf{0} + \mathrm{O}_{x,y,z}(5) \qquad \Longleftrightarrow \qquad H^3 \cong S^2 \times \mathbb{C}^1.$$

• **Proposition.** *After affine pre-normalizations :*

$$u = \frac{x^2}{2} + \frac{x^2y}{2} + F_{3,1,0}\,\frac{x^3y}{6} + \frac{x^2y^2}{2} + \boxed{F_{3,0,1}\,\frac{x^3z}{6}} + \mathrm{O}_{x,y,z}(5),$$

$$u' = \frac{x'^2}{2} + \frac{x'^2y'}{2} + F'_{3,1,0}\,\frac{x'^3y'}{6} + \frac{x'^2y'^2}{2} + \boxed{F'_{3,0,1}\,\frac{x'^3z'}{6}} + \mathrm{O}_{x',y',z'}(5),$$

there appears a relative invariant :

$$F'_{3,0,1} = \mathsf{nonzero} \cdot F_{3,0,1}.$$

• **Precisely :** [After group reductions]

$$\begin{bmatrix} x' \\ y' \\ z' \\ u' \end{bmatrix} = \begin{bmatrix} \mathsf{a} & \mathbf{0} & \mathbf{0} & -\mathsf{a}\,\mathsf{e} \\ \mathsf{e} & 1 & \mathbf{0} & \mathsf{h} \\ \mathsf{i} & \mathsf{j} & \mathsf{k} & \mathsf{l} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathsf{a}^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$$

$$F'_{3,0,1} = \frac{1}{\mathsf{a}\,\mathsf{k}}\,F_{3,0,1}.$$

• **Normalize :**

$$F_{3,0,1} := 1.$$

• **Proposition.** *After further normalizations :*

$$u = \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^3 z}{6} + \frac{x^2 y^2}{2} + F_{4,1,0}\,\frac{x^4 y}{24} + \frac{x^2 y^3}{2} + \frac{x^3 y z}{2} + \mathrm{O}_{x,y,z}(6).$$

• **Next :**

$F_{6,0,0}$ and $F_{5,1,0}$ are relative invariants.

• **Theorem.** *Only the branch* $F_{6,0,0} = 0$ *leads to a homogeneous model :*

$$u = \frac{1}{3}\,\frac{1}{z^2}\Big(-3\,y^2 - 1 + y^3 + 3\,xz + 3\,y - 3\,xyz + \sqrt{(1 - 2\,xz - 2\,y + y^2)^3}\Big)$$

or equivalently :

$$\Big(3\,z^2 u + 1 - 3\,y - 3\,xz + 3\,y^2 + 3\,xyz - y^3\Big)^2 - \big(1 - 2\,xz - 2\,y + y^2\big)^3.$$

with 1D isotropy :

$$\begin{bmatrix} \mathsf{a} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\mathsf{a}} & 0 \\ 0 & 0 & 0 & \mathsf{a}^2 \end{bmatrix}.$$

An Inexistence Phenomenon

Let $n \geqslant 1$, let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $v \in \mathbb{R}$. We consider equivalences of local analytic hypersurfaces $H^n \subset \mathbb{R}^{n+1}_{x,u}$ and $K^n \subset \mathbb{R}^{n+1}_{y,v}$ graphed as :

$$u = F(x_1, \dots, x_n) \qquad \text{and} \qquad v = G(y_1, \dots, y_n),$$

under *affine transformations* $\Psi \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$:

$$\begin{aligned} y_1 &= a_{1,1}\, x_1 + \cdots + a_{1,n}\, x_n + b_1\, u + \tau_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= a_{n,1}\, x_1 + \cdots + a_{n,n}\, x_n + b_n\, u + \tau_n, \\ v &= c_1\, x_1 + \cdots + c_n\, x_n + d\, u + \tau_0, \end{aligned}$$

where the $(n+1) \times (n+1)$ linear-part $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ matrix is invertible, *i.e.* belongs to $\mathsf{GL}(n+1, \mathbb{R})$. The collection of all these transformations is the Lie transformation group $\mathsf{Aff}(\mathbb{R}^{n+1})$.

The Lie algebra $\mathfrak{aff}(\mathbb{R}^{n+1})$ of $\mathsf{Aff}(\mathbb{R}^{n+1})$ consists of the vector fields :

$$
\begin{aligned}
L = {} & T_1 \frac{\partial}{\partial x_1} + \cdots + T_n \frac{\partial}{\partial x_n} + T_0 \frac{\partial}{\partial u} + \\
& + \Big(A_{1,1}\, x_1 + \cdots + A_{1,n}\, x_n + B_1\, u \Big) \frac{\partial}{\partial x_1} + \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
& + \Big(A_{n,1}\, x_1 + \cdots + A_{n,n}\, x_n + B_n\, u \Big) \frac{\partial}{\partial x_n} + \\
& + \Big(C_1\, x_1 + \cdots + C_n\, x_n + D\, u \Big) \frac{\partial}{\partial u}.
\end{aligned}
$$

A fixed hypersurface $H^n \subset \mathbb{R}^{n+1}$ has affine symmetry group the *local Lie group* :

$$
\mathsf{Sym}(H) := \big\{ \Psi \in \mathsf{Aff}(\mathbb{R}^{n+1}) \colon \ \Psi(H) \subset H \big\},
$$

where "$\subset$" is understood up to shrinking H, and where the transformations Ψ are close to the identity. Then $\mathsf{Sym}\, H$ has Lie algebra :

$$
\mathsf{Lie}\, \mathsf{Sym}(H) = \mathfrak{sym}(H) := \big\{ L \colon \ L\big|_H \text{ tangent to } H \big\}.
$$

Since all our considerations will be *local*, we can assume that everything takes place in some neighborhood of a fixed point $p_0 \in H$. Neighborhood shrinking is allowed (a finite number of times).

Definition. The hypersurface H is said to be *(locally) affinely homogeneous* if :

$$T_{p_0}H = \mathsf{Span}_{\mathbb{R}} \big\{ L\big|_{p_0} \colon L \in \mathfrak{sym}(H) \big\}.$$

According to Lie theory, the 1-parameter groups $p \longmapsto \mathsf{exp}(t\,L)(p)$ stabilize H, and $\mathsf{Sym}(H)$ is then locally transitive in a neighborhood of $p_0 \in H$.

The problem of classifying all affinely homogeneous n-dimensional local analytic smooth submanifolds $H^n \subset \mathbb{R}^{n+c}$ is probably of infinite complexity. Even for $n = 2 = c$, it is not terminated.

In the hypersurface case $c = 1$ and in dimension $n = 2$, the classification was terminated two decades ago [Abdalla-Dillen-Vrancken-1997, Doubrov-Komrakov-Rabinovich-1996, Eastwood-Ezhov-1999], cf. also [Arnaldsson-Valiquette-2020, Chen-Merker-2020] for a differential invariants perspective.

In dimension $n = 3$, and codimension $c = 1$, there is [Doubrov-Komrakov-1998] in which all *multiply transitive* models were classified, while for the *special* affine subgroup $\mathsf{Saff}(\mathbb{R}^{3+1}) \subset \mathsf{Aff}(\mathbb{R}^{3+1})$, there are the (unpublished) Ph.D. thesis of Marc Wermann [Wermann-2001], and the works of Eastwood-Ezhov [Eastwood-Ezhov-2001, Eastwood-Ezhov-2001-2], both complete.

Joint with Chen [Chen-Merker-2019], the author has studied the so-called *parabolic surfaces* $H^2 \subset \mathbb{R}^3$, those whose Hessian has constant rank 1 (*see* also [Chen-Merker-2020]).

Problem. *Study algebras of differential invariants and classify homogenous models of constant Hessian rank* 1 *hypersurfaces* $H^n \subset \mathbb{R}^{n+1}$.

A similar problem can be formulated in the context of CR geometry.

In Winter 2021, using a computer, all affinely homogenous Hessian rank 1 hypersurfaces $H^n \subset \mathbb{R}^{n+1}$ in dimensions $n = 2$, 3, 4, were determined.

Then exploring dimensions $n = 5, 6, 7$, it was surprising to realize that there are *no* homogenous models, except the degenerate ones obtained by taking a product of $\mathbb{R}^m$ with a homogeneous hypersurface $H^{n-m} \subset \mathbb{R}^{n-m+1}$ so that $2 \leqslant n - m \leqslant 4$.

To prove a *non-existence* result, which, incidentally, provides a complete classification (in the constant Hessian rank 1 branch), appeared to be unexpectedly hard. But the computational task appeared to be unexpectedly hard, and it took one year to write a detailed proof in general dimension $n \geqslant 5$.

Theorem. *Let $H^n \subset \mathbb{R}^{n+1}$ be a local affinely homogeneous hypersurface having constant Hessian rank 1. Then there exists an integer $1 \leqslant n_H \leqslant n$ and affine coordinates $(x_1, \ldots, x_n)$ in which :*

$$H^n = H^{n_H} \times \mathbb{R}^{n-n_H-1}_{x_{n_H+1},\ldots,x_n}$$

is a product of an affinely homogeneous hypersurface $H^{n_H} \subset \mathbb{R}^{n_H+1}$ times a 'dumb' $\mathbb{R}^{n-n_H-1}$, and is graphed as :

$$u = \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{n_H} \Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \tfrac{1}{2} \frac{x_i x_j}{(i-1)!(j-1)!} + \mathrm{O}_{x_2,\ldots,x_{m-1}}(3) \Big)$$
$$+ \sum_{m=n_H+2}^{\infty} E^m(x_1, \ldots, x_{n_H}),$$

with graphing function $F = F(x_1, \ldots, x_{n_H})$ independent of $x_{n_H+1}, \ldots, x_n$.

With $n_H = n$, this theorem shows the graphing function up to order $n+1$ included. Up to order $n+3$ included, we prove

Theorem. *In any dimension $n \geqslant 2$, every local hypersurface $H^n \subset \mathbb{R}^{n+1}$ having constant Hessian rank 1, and which is not affinely equivalent to a product of $\mathbb{R}^m$ $(1 \leqslant m \leqslant n)$ with a hypersurface $H^{n-m} \subset \mathbb{R}^{n-m+1}$, can be affinely normalized up to order $n+3$ as :*

$$
\begin{aligned}
u = \frac{x_1^2}{2} + \frac{x_1^2\,x_2}{2} + \sum_{m=3}^{n} \Big(\frac{x_1^m\,x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!} \Big) \\
+ F_{n+1,10\cdots 0}\,\frac{x_1^{n+1} x_2}{(n+1)!} + x_1^n \sum_{\substack{i,j \geqslant 2 \\ i+j=n+2}} \tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!} \\
+ F_{n+3,0\cdots 0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+2,10\cdots 0}\,\frac{x_1^{n+2}\,x_2}{(n+2)!} + F_{n+2,0010\cdots 0}\,\frac{x_1^{n+2}\,x_4}{(n+2)!} + \cdots + F_{n+2,0\cdots 01}\,\frac{x_1^{n+2}\,x_n}{(n+2)!} \\
+ F_{n+1,10\cdots 0}\,\frac{x_1^{n+1}\,x_2\,x_2}{n!} + x_1^{n+1} \sum_{\substack{i,j \geqslant 2 \\ i+j=n+3}} \tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!} \\
+ \mathrm{O}_{x_2,\dots,x_n}(3) + \mathrm{O}_{x_1,x_2,\dots,x_n}(n+4).
\end{aligned}
$$

Furthermore, linear $\mathsf{GL}(n+1,\mathbb{R})$ self-maps (fixing the origin) $\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$ of such a hypersurface are necessarily weighted dilations of the form, with $\mathbf{c} \in \mathbb{R}^*$:

$$
y_1 = \mathbf{c}\,x_1, \quad y_2 = 0, \quad y_3 = \frac{1}{\mathbf{c}}\,x_3, \quad \dots, \quad y_n = \frac{1}{\mathbf{c}^{n-2}}\,x_n, \quad v = \mathbf{c}^2\,u,
$$

- In dimension $n = 2$:

$$u = \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + F_{4,0}\,\frac{x_1^3 x_1}{24} + F_{3,1}\,\frac{x_1^3 x_2}{6},$$

the stability group is :

$$G^3_{\mathsf{stab}}: \qquad \begin{bmatrix} a_{1,1} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & 1 & b_2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{3}},$$

and its action gives :

$$0 \overset{40}{=} -\frac{1}{24}\,a_{1,1}^2\,F_{4,0} + \frac{1}{24}\,a_{1,1}^4\,G_{4,0} + \frac{1}{6}\,a_{1,1}^3\,a_{2,1}\,G_{3,1} + \frac{1}{8}\,a_{1,1}^2\,a_{2,1}^2 + \frac{1}{4}\,a_{1,1}^2\,\boxed{b_2},$$

$$0 \overset{31}{=} -\frac{1}{6}\,a_{1,1}^2\,F_{3,1} + \frac{1}{6}\,a_{1,1}^3\,G_{3,1}.$$

The free group parameter b_2 can be used to normalize $G_{4,0} := 0$.

- In dimension $n = 3$:

$$\begin{aligned} u \; = \; & \frac{x_1^2}{2} \\ & + \frac{x_1^2 x_2}{2} \\ & + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} \\ & + F_{5,0,0} \, \frac{x_1^4 x_1}{120} + F_{4,1,0} \, \frac{x_1^4 x_2}{24} + F_{4,0,1} \, \frac{x_1^4 x_3}{24} + \frac{x_1^3 x_2 x_3}{2}, \end{aligned}$$

the stability group is :

$$G^4_{\mathsf{stab}} : \quad \begin{bmatrix} a_{1,1} & \mathbf{0} & \mathbf{0} & -a_{1,1} a_{2,1} \\ a_{2,1} & 1 & \mathbf{0} & -\frac{1}{2} a_{2,1}^2 - \frac{2}{3} a_{1,1} a_{3,1} \\ a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & b_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{4}},$$

and its action gives :

$$0 \overset{500}{=} -\frac{1}{120}\,a_{1,1}^2\,F_{5,0,0} + \frac{1}{120}\,a_{1,1}^5\,G_{5,0,0} + \frac{1}{24}\,a_{1,1}^4\,a_{2,1}\,G_{4,1,0} + \frac{1}{12}\,a_{1,1}^3\,a_{2,1}\,a_{3,1} + \frac{1}{24}\,a_{1,1}^4\,a_{3,1}\,G_{4,0,1} + \frac{1}{12}\,a_{1,1}^3\,\boxed{b_3}$$

$$0 \overset{410}{=} -\frac{1}{24}\,a_{1,1}^2\,F_{4,1,0} + \frac{1}{24}\,a_{1,1}^4\,G_{4,1,0},$$

$$0 \overset{401}{=} -\frac{1}{24}\,a_{1,1}^2\,F_{4,0,1} + \frac{1}{24}\,a_{1,1}^3\,G_{4,0,1} + \frac{1}{12}\,a_{1,1}^2\,\boxed{a_{2,1}}.$$

The free group parameter $a_{2,1}$ can be used to normalize $G_{4,0,1} := 0$, and the free group parameter b_3 can be used to normalize $G_{5,0,0} := 0$.

- In dimension $n = 4$:

$$\begin{aligned} u = {} & \frac{x_1^2}{2} \\ & + \frac{x_1^2 x_2}{2} \\ & + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} \\ & + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2} \\ & + F_{6,0,0,0}\,\frac{x_1^5 x_1}{720} + F_{5,1,0,0}\,\frac{x_1^5 x_2}{120} + F_{5,0,1,0}\,\frac{x_1^5 x_3}{120} + F_{5,0,0,1}\,\frac{x_1^5 x_4}{120} + \frac{x_1^2 x_2^4}{2} + x_1^3 x_2^2 x_3 + \frac{x_1^4 x_2 x_4}{6} + \frac{x_1^4 x_3^2}{8}, \end{aligned}$$

the stability group is :

$$G^5_{\mathsf{stab}}: \quad \begin{bmatrix} a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & 1 & \mathbf{0} & \mathbf{0} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{1,1}a_{3,1} \\ a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} & -\frac{1}{2}a_{1,1}a_{4,1} - a_{2,1}a_{3,1} \\ a_{4,1} & \mathbf{0} & \frac{-2a_{2,1}}{a_{1,1}^2} & \frac{1}{a_{1,1}^2} & b_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{5}},$$

and its action gives :

$$\begin{aligned} 0 &\overset{6000}{=} -\frac{1}{720}\, a_{1,1}^2\, F_{6,0,0,0} + \frac{1}{720}\, a_{1,1}^6\, G_{6,0,0,0} + \frac{1}{72}\, a_{1,1}^4\, a_{3,1}^2 + \frac{1}{48}\, a_{1,1}^4\, a_{2,1}\, a_{4,1} \\ &\qquad + \frac{1}{120}\, G_{5,1,0,0}\, a_{1,1}^5\, a_{2,1} + \frac{1}{120}\, G_{5,0,1,0}\, a_{1,1}^5\, a_{3,1} + \frac{1}{120}\, G_{5,0,0,1}\, a_{1,1}^5\, a_{4,1} + \frac{1}{48}\, a_{1,1}^4\, \boxed{b_4} \\ 0 &\overset{5100}{=} -\frac{1}{120}\, a_{1,1}^2\, F_{5,1,0,0} + \frac{1}{120}\, a_{1,1}^5\, G_{5,1,0,0}, \\ 0 &\overset{5010}{=} -\frac{1}{120}\, a_{1,1}^2\, F_{5,0,1,0} + \frac{1}{120}\, a_{1,1}^4\, G_{5,0,1,0} - \frac{1}{24}\, a_{1,1}^2\, a_{2,1}^2 - \frac{1}{60}\, a_{1,1}^3\, a_{2,1} + \frac{1}{36}\, a_{1,1}^3\, \boxed{a_{3,1}}, \\ 0 &\overset{5001}{=} -\frac{1}{120}\, a_{1,1}^2\, F_{5,0,0,1} + \frac{1}{120}\, a_{1,1}^3\, G_{5,0,0,1} + \frac{1}{24}\, a_{1,1}^2\, \boxed{a_{2,1}}. \end{aligned}$$

The free group parameters $a_{2,1}$, $a_{3,1}$, b_4 can be used to normalize $G_{5,0,0,1} := 0$, $G_{5,0,1,0} := 0$, $G_{6,0,0,0} := 0$.

- In dimension $n = 5$:

$$
\begin{aligned}
u \;=\; & \frac{x_1^2}{2} \\
& + \frac{x_1^2 x_2}{2} \\
& + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} \\
& + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2} \\
& + \frac{x_1^5 x_5}{120} + \frac{x_1^2 x_2^4}{2} + x_1^3 x_2^2 x_3 + \frac{x_1^4 x_2 x_4}{6} + \frac{x_1^4 x_3^2}{8} \\
& + F_{7,0,0,0,0}\,\frac{x_1^6 x_1}{5040} + F_{6,1,0,0,0}\,\frac{x_1^6 x_2}{720} + F_{6,0,1,0,0}\,\frac{x_1^6 x_3}{720} + F_{6,0,0,1,0}\,\frac{x_1^6 x_4}{720} + F_{6,0,0,0,1}\,\frac{x_1^6 x_5}{720} \\
& + \tfrac{1}{2}\, x_1^2 x_2^5 + \tfrac{5}{3}\, x_1^3 x_2^3 x_3 + \tfrac{5}{8}\, x_1^4 x_2 x_3^2 + \tfrac{5}{12}\, x_1^4 x_2^2 x_4 + \tfrac{1}{12}\, x_1^5 x_3 x_4 + \tfrac{1}{24}\, x_1^5 x_2 x_5,
\end{aligned}
$$

the stability group is :

$$G^6_{\mathsf{stab}}: \begin{bmatrix} a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{1,1}a_{3,1} \\ a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} & \mathbf{0} & -a_{2,1}a_{3,1} - \frac{1}{2}a_{1,1}a_{4,1} \\ a_{4,1} & \mathbf{0} & -\frac{2a_{2,1}}{a_{1,1}^2} & \frac{1}{a_{1,1}^2} & \mathbf{0} & -a_{2,1}a_{4,1} - \frac{2}{3}a_{3,1}^2 - \frac{2}{5}a_{1,1}a_{5,1} \\ a_{5,1} & \mathbf{0} & \frac{5a_{2,1}^2}{a_{1,1}^3} - \frac{10}{3}\frac{a_{3,1}}{a_{1,1}^2} & -\frac{5a_{2,1}}{a_{1,1}^3} & \frac{1}{a_{1,1}^3} & b_5 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{6}},$$

and its action gives :

$$\begin{aligned}
0 &\overset{70000}{=} -\frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,0,0} + \frac{1}{5040}\,a_{1,1}^7\,G_{7,0,0,0,0} + \frac{1}{144}\,a_{1,1}^5\,a_{3,1}\,a_{4,1} + \frac{1}{240}\,a_{1,1}^5\,a_{2,1}\,a_{5,1} \\
&\quad + \frac{1}{720}\,G_{6,1,0,0,0}\,a_{1,1}^6\,a_{2,1} + \frac{1}{720}\,G_{6,0,1,0,0}\,a_{1,1}^6\,a_{3,1} + \frac{1}{720}\,G_{6,0,0,1,0}\,a_{1,1}^6\,a_{4,1} + \frac{1}{720}\,G_{6,0,0,0,1}\,a_{1,1}^6\,a_{5,1} + \frac{1}{240}\,a_{1,1}^5\,\boxed{b_5} \\
0 &\overset{61000}{=} -\frac{1}{720}\,a_{1,1}^2\,F_{6,1,0,0,0} + \frac{1}{720}\,a_{1,1}^6\,G_{6,1,0,0,0}, \\
0 &\overset{60100}{=} -\frac{1}{720}\,a_{1,1}^2\,F_{6,0,1,0,0} + \frac{1}{720}\,a_{1,1}^5\,G_{6,0,1,0,0} - \frac{1}{36}\,a_{1,1}^3\,a_{2,1}\,a_{3,1} + \frac{1}{48}\,a_{1,1}^2\,a_{2,1}^3 + \frac{1}{144}\,a_{1,1}^3\,a_{2,1}^2\,G_{6,0,0,0,1} \\
&\quad - \frac{1}{216}\,a_{1,1}^4\,a_{3,1}\,G_{6,0,0,0,1} - \frac{1}{360}\,a_{1,1}^4\,a_{2,1}\,G_{6,0,0,1,0} + \frac{1}{144}\,a_{1,1}^4\,\boxed{a_{4,1}}, \\
0 &\overset{60010}{=} -\frac{1}{720}\,a_{1,1}^2\,F_{6,0,0,1,0} + \frac{1}{720}\,a_{1,1}^4\,G_{6,0,0,1,0} - \frac{1}{32}\,a_{1,1}^2\,a_{2,1}^2 - \frac{1}{144}\,a_{1,1}^3\,a_{2,1}\,G_{6,0,0,0,1} + \frac{1}{72}\,a_{1,1}^3\,\boxed{a_{3,1}}, \\
0 &\overset{60001}{=} -\frac{1}{720}\,a_{1,1}^2\,F_{6,0,0,0,1} + \frac{1}{720}\,a_{1,1}^3\,G_{6,0,0,0,1} + \frac{1}{80}\,a_{1,1}^2\,\boxed{a_{2,1}}.
\end{aligned}$$

The free group parameters $a_{2,1}$, $a_{3,1}$, $a_{4,1}$, b_4 can be used to normalize $G_{6,0,0,0,1} := 0$, $G_{6,0,0,1,0} := 0$, $G_{6,0,1,0,0} := 0$, $G_{7,0,0,0,0} := 0$.

- In dimension $n = 6$:

$$
\begin{aligned}
u \;=\; & \frac{x_1^2}{2} \\
& + \frac{x_1^2 x_2}{2} \\
& + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} \\
& + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2} \\
& + \frac{x_1^5 x_5}{120} + \frac{x_1^2 x_2^4}{2} + x_1^3 x_2^2 x_3 + \frac{x_1^4 x_2 x_4}{6} + \frac{x_1^4 x_3^2}{8} \\
& + \frac{x_1^6 x_6}{720} + \frac{x_1^2 x_2^5}{2} + \tfrac{5}{3}\, x_1^3 x_2^3 x_3 + \tfrac{5}{8}\, x_1^4 x_2 x_3^2 + \tfrac{5}{12}\, x_1^4 x_2^2 x_4 + \tfrac{1}{12}\, x_1^5 x_3 x_4 + \tfrac{1}{24}\, x_1^5 x_2 x_5 \\
& + F_{8,0,0,0,0,0}\, \frac{x_1^7 x_1}{40320} + F_{7,1,0,0,0,0}\, \frac{x_1^7 x_2}{5040} + F_{7,0,1,0,0,0}\, \frac{x_1^7 x_3}{5040} + F_{7,0,0,1,0,0}\, \frac{x_1^7 x_4}{5040} + F_{7,0,0,0,1,0}\, \frac{x_1^7 x_5}{5040} + F_{7,0,0,0,0,1}\, \frac{x_1^7 x_6}{5040} \\
& + \tfrac{1}{8}\, x_1^5 x_3^3 + \tfrac{1}{72}\, x_1^6 x_4^2 + \tfrac{5}{2}\, x_1^3 x_2^4 x_3 + \tfrac{15}{8}\, x_1^4 x_2^2 x_3^2 + \tfrac{5}{6}\, x_1^4 x_2^3 x_4 + \tfrac{1}{8}\, x_1^5 x_2^2 x_5 + \tfrac{1}{48}\, x_1^6 x_3 x_5 + \tfrac{1}{2}\, x_1^5 x_2 x_3 x_4 + \tfrac{1}{2}\, x_1^2 x_2^6 + \tfrac{1}{120}\, x_1^6 x_2 x_6,
\end{aligned}
$$

the stability group is :

$$
\begin{bmatrix}
a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{1,1} a_{2,1} \\
a_{2,1} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2} a_{2,1}^2 - \frac{2}{3} a_{1,1} a_{3,1} \\
a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{2,1} a_{3,1} - \frac{1}{2} a_{1,1} a_{4,1} \\
a_{4,1} & \mathbf{0} & -\frac{2 a_{2,1}}{a_{1,1}^2} & \frac{1}{a_{1,1}^2} & \mathbf{0} & \mathbf{0} & -a_{2,1} a_{4,1} - \frac{2}{3} a_{3,1}^2 - \frac{2}{5} a_{1,1} a_{5,1} \\
a_{5,1} & \mathbf{0} & \frac{5 a_{2,1}^2}{a_{1,1}^3} - \frac{10}{3} \frac{a_{3,1}}{a_{1,1}^2} & -\frac{5 a_{2,1}}{a_{1,1}^3} & \frac{1}{a_{1,1}^3} & \mathbf{0} & -a_{2,1} a_{5,1} - \frac{1}{3} a_{1,1} a_{6,1} - \frac{5}{3} a_{3,1} a_{4,1} \\
a_{6,1} & \mathbf{0} & 20 \frac{a_{2,1} a_{3,1}}{a_{1,1}^3} - 15 \frac{a_{2,1}^3}{a_{1,1}^4} - 5 \frac{a_{4,1}}{a_{2,1}^2} & \frac{45}{2} \frac{a_{2,1}^2}{a_{1,1}^4} - 10 \frac{a_{3,1}}{a_{1,1}^3} & -9 \frac{a_{2,1}}{a_{1,1}^4} & \frac{1}{a_{1,1}^4} & b_6 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2
\end{bmatrix}^{7},
$$

and its action gives :

$$
\begin{aligned}
0 &\overset{800000}{=} -\frac{1}{40320}\,a_{1,1}^2\,F_{8,0,0,0,0,0}+\frac{1}{40320}\,a_{1,1}^7\,G_{8,0,0,0,0,0}+\\
&+\frac{1}{5040}\,G_{7,1,0,0,0,0}\,a_{1,1}^7\,a_{2,1}+\frac{1}{5040}\,G_{7,0,1,0,0,0}\,a_{1,1}^7\,a_{3,1}+\frac{1}{5040}\,G_{7,0,0,1,0,0}\,a_{1,1}^7\,a_{4,1}+\frac{1}{5040}\,G_{7,0,0,0,1,0}\,a_{1,1}^7\,a_{5,1}+\frac{1}{5040}\,G_{7,0,0,0,0,1}\,a_{1,1}^7\,a_{6,1}\\
&+\frac{1}{1152}\,a_{1,1}^6\,a_{4,1}^2+\frac{1}{1440}\,a_{1,1}^6\,a_{2,1}\,a_{6,1}+\frac{1}{720}\,a_{1,1}^6\,a_{3,1}\,a_{5,1}+\frac{1}{1440}\,a_{1,1}^6\,\boxed{b_6}\\
0 &\overset{710000}{=} -\frac{1}{5040}\,a_{1,1}^2\,F_{7,1,0,0,0,0}+\frac{1}{5040}\,a_{1,1}^7\,G_{7,1,0,0,0,0},\\
0 &\overset{701000}{=} -\frac{1}{5040}\,a_{1,1}^2\,F_{7,0,1,0,0,0}+\frac{1}{5040}\,a_{1,1}^6\,G_{7,0,1,0,0,0}-\frac{1}{144}\,a_{1,1}^4\,a_{2,1}\,a_{4,1}+\frac{1}{48}\,a_{1,1}^3\,a_{2,1}^2\,a_{3,1}-\frac{1}{96}\,a_{1,1}^2\,a_{2,1}^4-\frac{1}{216}\,a_{1,1}^4\,a_{3,1}^2\\
&-\frac{1}{2520}\,a_{1,1}^5\,a_{2,1}\,G_{7,0,0,1,0,0}+\frac{1}{252}\,a_{1,1}^4\,a_{2,1}\,a_{3,1}\,G_{7,0,0,0,0,1}-\frac{1}{336}\,a_{1,1}^3\,a_{2,1}^3\,G_{7,0,0,0,0,1}-\frac{1}{1008}\,a_{1,1}^4\,a_{4,1}\,G_{7,0,0,0,0,1}-\frac{1}{1008}\,a_{1,1}^4\,a_{2,1}^2\,G_{7,0,0,0,1,0}\\
&-\frac{1}{1512}\,a_{1,1}^5\,a_{3,1}\,G_{7,0,0,0,1,0}+\frac{1}{720}\,a_{1,1}^5\,\boxed{a_{5,1}},\\
0 &\overset{700100}{=} -\frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,1,0,0}+\frac{1}{5040}\,a_{1,1}^5\,G_{7,0,0,1,0,0}+\frac{1}{224}\,a_{1,1}^3\,a_{2,1}^2\,G_{7,0,0,0,0,1}-\frac{1}{504}\,a_{1,1}^4\,a_{3,1}\,G_{7,0,0,0,0,1}+\frac{1}{48}\,a_{1,1}^2\,a_{2,1}^3\\
&-\frac{1}{48}\,a_{1,1}^3\,a_{2,1}\,a_{3,1}-\frac{1}{1008}\,a_{1,1}^4\,a_{2,1}\,G_{7,0,0,0,1,0}+\frac{1}{288}\,a_{1,1}^4\,\boxed{a_{4,1}},\\
0 &\overset{700010}{=} -\frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,0,1,0}+\frac{1}{5040}\,a_{1,1}^4\,G_{7,0,0,0,1,0}-\frac{1}{560}\,a_{1,1}^3\,a_{2,1}\,G_{7,0,0,0,0,1}-\frac{1}{80}\,a_{1,1}^2\,a_{2,1}^2+\frac{1}{240}\,a_{1,1}^3\,\boxed{a_{3,1}},\\
0 &\overset{700001}{=} -\frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,0,0,1}+\frac{1}{5040}\,a_{1,1}^3\,G_{7,0,0,0,0,1}+\frac{1}{360}\,a_{1,1}^2\,\boxed{a_{2,1}}.
\end{aligned}
$$

The free group parameters $a_{2,1}$, $a_{3,1}$, $a_{4,1}$, $a_{5,1}$, b_4 can be used to normalize $G_{7,0,0,0,0,1} := 0$, $G_{7,0,0,0,1,0} := 0$, $G_{7,0,0,1,0,0} := 0$, $G_{7,0,1,0,0,0} := 0$, $G_{8,0,0,0,0,0} := 0$.

Instead of attempting to *dominate* the combinatorics of such formulas in any dimension $n \geqslant 2$, we will *infinitesimalize* the determination of the stability group at order $n+1$, and also, we will *infinitesimalize* its action on coefficients of order $n+2$.

The following more advanced theorem gives terms of orders $n+4$, $n+5$, which are more complicated, but (unfortunately) necessary in order to establish the (unexpected) main inexistence result.

Theorem. *In any dimension $n \geqslant 2$, every local hypersurface $H^n \subset \mathbb{R}^{n+1}$ having constant Hessian rank 1 which is not affinely equivalent to a product of $\mathbb{R}^m$ $(1 \leqslant m \leqslant n)$ with a hypersurface $H^{n-m} \subset \mathbb{R}^{n-m+1}$ can be affinely normalized as :*

$$
\begin{aligned}
u = \frac{x_1^2}{2} + \frac{x_1^2\,x_2}{2} + \sum_{m=3}^{n} \bigg(\frac{x_1^m\,x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!}\bigg)
\\
+ F_{n+1,10\cdots 0}\,\frac{x_1^{n+1}x_2}{(n+1)!} + x_1^n \sum_{\substack{i,j\geqslant 2\\ i+j=n+2}} \tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!}
\\
+ F_{n+3,0\cdots 0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+2,10\cdots 0}\,\frac{x_1^{n+2}\,x_2}{(n+2)!} + F_{n+2,0010\cdots 0}\,\frac{x_1^{n+2}\,x_4}{(n+2)!} + \cdots + F_{n+2,0\cdots 01}\,\frac{x_1^{n+2}\,x_n}{(n+2)!}
\\
+ F_{n+1,10\cdots 0}\,\frac{x_1^{n+1}\,x_2\,x_2}{n!} + x_1^{n+1} \sum_{\substack{i,j\geqslant 2\\ i+j=n+3}} \tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!}
\end{aligned}
$$

$$
\begin{aligned}
&+F_{n+4,0\cdots0}\,\frac{x_1^{n+4}}{(n+4)!}+F_{n+3,10\cdots0}\,\frac{x_1^{n+3}\,x_2}{(n+3)!}+F_{n+3,010\cdots0}\,\frac{x_1^{n+3}\,x_3}{(n+3)!}+F_{n+3,0010\cdots0}\,\frac{x_1^{n+3}\,x_4}{(n+3)!}+\cdots+F_{n+3,0\cdots01}\,\frac{x_1^{n+3}\,x_n}{(n+3)!}\\
&\qquad+x_1^{n+2}\bigg[\frac{F_{n+2,10\cdots0}}{(n+1)!}\,x_2\,x_2+\frac{F_{n+1,10\cdots0}}{2!\,n!}\,x_2\,x_3+\frac{F_{n+2,0010\cdots0}}{(n+1)!}\,x_2\,x_4+\cdots+\frac{F_{n+2,0\cdots01}}{(n+1)!}\,x_2\,x_n\\
&\qquad\qquad+\sum_{\substack{i,j\geqslant 2\\ i+j=n+4}}\tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!}\bigg]\\
&+F_{n+5,0\cdots0}\,\frac{x_1^{n+5}}{(n+5)!}+F_{n+4,10\cdots0}\,\frac{x_1^{n+4}\,x_2}{(n+4)!}+F_{n+4,010\cdots0}\,\frac{x_1^{n+4}\,x_3}{(n+4)!}+F_{n+4,0010\cdots0}\,\frac{x_1^{n+4}\,x_4}{(n+4)!}+\cdots+F_{n+4,0\cdots01}\,\frac{x_1^{n+4}\,x_n}{(n+4)!}\\
&+x_1^{n+3}\bigg[\bigg(\frac{F_{n+3,10\cdots0}}{(n+2)!}-\frac{F_{n+3,0\cdots0}}{2!\,(n+1)!}\bigg)x_2x_2+\bigg(\frac{F_{n+3,010\cdots0}}{(n+2)!}+\frac{F_{n+2,10\cdots0}}{2!\,(n+1)!}\bigg)x_2x_3+\bigg(\frac{F_{n+3,0010\cdots0}}{(n+2)!}+\frac{F_{n+1,10\cdots0}}{3!\,n!}\bigg)x_2x_4\\
&\qquad+\frac{F_{n+3,00010\cdots0}}{(n+2)!}\,x_2x_5+\cdots+\frac{F_{n+3,0\cdots01}}{(n+2)!}\,x_2x_n\\
&\qquad+\frac{F_{n+2,0010\cdots0}}{2!\,(n+2)!}\,x_3x_4+\cdots+\frac{F_{n+2,0\cdots01}}{2!\,(n+2)!}\,x_3x_n\\
&\qquad\qquad+\sum_{\substack{i,j\geqslant 2\\ i+j=n+5}}\tfrac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!}\bigg]+\mathrm{O}_{x_2,\dots,x_n}(3)+\mathrm{O}_{x_1,x_2,\dots,x_n}(n+6).
\end{aligned}
$$

At order $n+5$, the isotropy is at most one-dimensional.

Main Theorem. *In any dimension $n \geqslant 5$, there are* no *affinely homogeneous constant Hessian rank 1 nondegenerate hypersurfaces :*

$$H^n \subset \mathbb{R}^{n+1}.$$

Summary of the Proof of the Main Theorem

Now, we take a general affine vector field which does not necessarily vanish at the origin :

$$
\begin{aligned}
L = & \left(T_0 + C_1\,x_1 + C_2\,x_2 + \cdots + C_{n-1}\,x_{n-1} + C_n\,x_n\right) \frac{\partial}{\partial u} \\
& + \left(T_1 + A_{1,1}\,x_1 + A_{1,2}\,x_2 + \cdots + A_{1,n-1}\,x_{n-1} + A_{1,n}\,x_n\right) \frac{\partial}{\partial x_1} \\
& + \left(T_2 + A_{2,1}\,x_1 + A_{2,2}\,x_2 + \cdots + A_{2,n-1}\,x_{n-1} + A_{2,n}\,x_n\right) \frac{\partial}{\partial x_2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \left(T_{n-1} + A_{n-1,1}\,x_1 + A_{n-1,2}\,x_2 + \cdots + A_{n-1,n-1}\,x_{n-1} + A_{n-1,n}\,x_n\right) \frac{\partial}{\partial x_{n-1}} \\
& + \left(T_n + A_{n,1}\,x_1 + A_{n,2}\,x_2 + \cdots + A_{n,n-1}\,x_{n-1} + A_{n,n}\,x_n\right) \frac{\partial}{\partial x_n}.
\end{aligned}
$$

Remind that if L is tangent to the hypersurface $H = \big\{u = F(x_1, \dots, x_n)\big\}$, then $T_0 = 0$, since $u = F = \mathrm{O}_x(2)$.

Here, the parameters $T_1, \dots, T_n$ are tightly related to the *infinitesimal transitivity* of the action, since the value of L at the origin is :

$$L\big|_0 = T_1 \frac{\partial}{\partial x_1} + \cdots + T_n \frac{\partial}{\partial x_n},$$

since homogeneity requires that :

$$T_0 H = \mathsf{Span}_{\mathbb{R}} \big\{ L\big|_0 \colon \ L|_H \text{ tangent to } H \big\},$$

and since, due to again to $F = \mathrm{O}_x(2)$:

$$T_0 H = \mathsf{Span} \left\{ \frac{\partial}{\partial x_1}, \ \dots, \ \frac{\partial}{\partial x_n} \right\}.$$

Observation. *For (infinitesimal) affine homogeneity to hold, the parameters $T_1, \dots, T_n$ should remain absolutely free in all computations.* □

Now, such a general affine vector field L is an infinitesimal affine symmetry of our hypersurface $H = \big\{ u = F(x_1, \dots, x_n) \big\}$ graphed as above, if and only if $L|_H$ is tangent to H, if and only if the following power series identity holds in $\mathbb{R}\{x_1, \dots, x_n\}$:

$$0 \equiv L\big(-u + F \big)\Big|_{u=F}.$$

We will in fact 'only' study independent monomials of order $\leqslant n+4$ in this fundamental equation, namely we will examine/compute :

$$\pi_{\mathsf{ind}}^{n+4}\Big(L\big(-\,u+F\big)\Big|_{u=F}\Big).$$

Reminding — *see* also below — that :

$$F_{x_1}\,=\,\mathrm{O}_x(1),\;\; F_{x_2}\,=\,\mathrm{O}_x(2),\;\; \cdots,\;\; F_{x_{n-1}}\,=\,\mathrm{O}_x(n-1),\;\; F_{x_n}\,=\,\mathrm{O}_x(n),$$

we may therefore start out by writing :

$$\begin{aligned}
0 \equiv & -C_1\,x_1 - C_2\,x_2 - \cdots - C_{n-1}\,x_{n-1} - C_n\,x_n - D\,\big[\pi_{\mathsf{ind}}^{n+4}(F)\big] && \Lambda_0 \\
& + \Big(T_1 + A_{1,1}\,x_1 + A_{1,2}\,x_2 + \cdots + A_{1,n-1}\,x_{n-1} + A_{1,n}\,x_n + B_1\,\big[\pi_{\mathsf{ind}}^{n+3}(F)\big]\Big)\,F_{x_1} && \Lambda_1 \\
& + \Big(T_2 + A_{2,1}\,x_1 + A_{2,2}\,x_2 + \cdots + A_{2,n-1}\,x_{n-1} + A_{2,n}\,x_n + B_2\,\big[\pi_{\mathsf{ind}}^{n+2}(F)\big]\Big)\,F_{x_2} && \Lambda_2 \\
& + \Big(T_3 + A_{3,1}\,x_1 + A_{3,2}\,x_2 + \cdots + A_{3,n-1}\,x_{n-1} + A_{3,n}\,x_n + B_3\,\big[\pi_{\mathsf{ind}}^{n+1}(F)\big]\Big)\,F_{x_3} && \Lambda_3 \\
& + \Big(T_4 + A_{4,1}\,x_1 + A_{4,2}\,x_2 + \cdots + A_{4,n-1}\,x_{n-1} + A_{4,n}\,x_n + B_4\,\big[\pi_{\mathsf{ind}}^{n}(F)\big]\Big)\,F_{x_4} && \Lambda_4 \\
& + \Big(T_5 + A_{5,1}\,x_1 + A_{5,2}\,x_2 + \cdots + A_{5,n-1}\,x_{n-1} + A_{5,n}\,x_n + B_5\,\big[\pi_{\mathsf{ind}}^{n-1}(F)\big]\Big)\,F_{x_5} && \Lambda_5 \\
& + \Big(T_6 + A_{6,1}\,x_1 + A_{6,2}\,x_2 + \cdots + A_{6,n-1}\,x_{n-1} + A_{6,n}\,x_n + B_6\,\big[\pi_{\mathsf{ind}}^{n-2}(F)\big]\Big)\,F_{x_6} && \Lambda_6 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && \cdots \\
& + \Big(T_{n-1} + A_{n-1,1}x_1 + A_{n-1,2}x_2 + \cdots + A_{n-1,n-1}x_{n-1} + A_{n-1,n}x_n + B_{n-1}\big[\pi_{\mathsf{ind}}^{5}(F)\big]\Big)F_{x_{n-1}} && \Lambda_{n-1} \\
& + \Big(T_n + A_{n,1}\,x_1 + A_{n,2}\,x_2 + \cdots + A_{n,n-1}\,x_{n-1} + A_{n,n}\,x_n + B_n\,\big[\pi_{\mathsf{ind}}^{4}(F)\big]\Big)\,F_{x_n}. && \Lambda_n
\end{aligned}$$

Notice on the right above that we attribute names to these $1+n$ lines :

$$\Lambda_0, \qquad \Lambda_1, \qquad \Lambda_2, \qquad \Lambda_3, \qquad \Lambda_4, \qquad \Lambda_5, \qquad \Lambda_6, \qquad \ldots, \qquad \Lambda_{n-1}, \qquad \Lambda_n.$$

In the paper, we compute some of the coefficients of the monomials in this large equation, namely we will compute some coefficients :

$$E_{[\sigma_1,\dots,\sigma_n]} := \big[x^{\sigma_1}\cdots x_n^{\sigma_n}\big]\Big(L\big(-\,u+F\big)\Big|_{u=F}\Big) \qquad (\sigma_1+\cdots+\sigma_n \leqslant n+4),$$

which are linear in $C_\bullet$, D, $T_\bullet$, $A_{\bullet,\bullet}$, $B_\bullet$, and which should vanish for L to really be tangent to H :

$$E_{[\sigma_1,\dots,\sigma_n]} = 0.$$

Especially, we compute :

$$\begin{aligned}\mathbf{I} &:= E_{[n+2,0,\dots,0,1]} = 0,\\ \mathbf{II} &:= E_{[n+3,0,\dots,0,1]} = 0.\end{aligned}$$

Here is the key reason why affinely homogeneous models do *not* exist in dimension $n \geqslant 5$. We use $*$ to denote any unspecified real number whose value does not matter.

Proposition. *For a hypersurface* $\{u = F(x)\}$ *normalized as above, after taking account of some of the other equations* $E_{[\sigma_1,\dots,\sigma_n]} = 0$*, these two* *specific equations* **I**, **II** *become of the form :*

$$0 \overset{\mathbf{I}}{=} *T_1 + *T_2 - \frac{1}{12\,(n-3)\,n!}\,T_4 + \frac{2}{(n+2)!}\,F_{n+2,0\cdots 01}\,A_{1,1},$$

$$0 \overset{\mathbf{II}}{=} *T_1 + *T_2 + *T_3 - \frac{1}{30\,(n-4)\,n!}\,T_5 + \frac{3}{(n+3)!}\,F_{n+3,0\cdots 01}\,A_{1,1}.$$

Admitting temporarily this fact, we can easily conclude the main *inexistence* result.

Proof of the Main Theorem. If the power series coefficient $F_{n+2,0\cdots 01} = 0$ would be zero, then the first equation :

$$0 \overset{\mathbf{I}}{=} *T_1 + *T_2 - \frac{1}{12\,(n-3)\,n!}\,T_4,$$

would consist of a *nontrivial* linear dependence relation between $T_1, \ldots, T_n$, contradicting infinitesimal transitivity.

So necessarily, $F_{n+2,0\cdots 01} \neq 0$.

But then from equation **I**, we can solve the isotropy parameter :

$$A_{1,1} = *T_1 + *T_2 + *T_4,$$

that we replace in **II**, getting, whatever the value of $F_{n+3,0\cdots 01}$ is :

$$0 = *T_1 + *T_2 + *T_3 + *T_4 - \frac{1}{30\,(n-4)\,n!}\,T_5.$$

But such an equation is also *always* a *nontrivial* linear dependence relation between $T_1, \ldots, T_n$, contradicting again infinitesimal transitivity !

Observe that $n \geqslant 5$ is used in this argumentation !

Handling Differential Invariants as Taylor Coefficients

- **Local surface $S^2 \subset \mathbb{C}^3$:**

$$u = F(x,y) = \sum_{j+k \geqslant 0} F_{j,k} \frac{x^j}{j!} \frac{y^k}{k!}.$$

- **Hypothesis :**

$$2 = \mathsf{rank} \begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix}$$

- **Elementary affine prenormalization :**

$$u = x\,y + G_{3,0} \frac{x^3}{6} + G_{0,3} \frac{y^3}{6} + \sum_{j+k \geqslant 4} G_{j,k} \frac{x^j}{j!} \frac{y^k}{k!}.$$

- **Pick invariant :**

$$\mathsf{Pick} \propto G_{3,0} \cdot G_{0,3}.$$

- **Elimination computation :** *All the* $G_{j,k} = G_{j,k}(F_{\bullet,\bullet})$ *express in terms of the* $F_{l,m}$ *with* $l + m \leqslant j + k$.
- **Example :**

$$
\begin{aligned}
G_{3,0} = & -\frac{1}{2\sqrt{2}(F_{2,0}+2F_{1,1}+F_{0,2})^{\frac{3}{2}}(F_{1,1}^2 - F_{2,0}F_{0,2})^{\frac{3}{2}}} \\
& \times \Big\{ -4F_{0,3}F_{1,1}^3 + 6F_{0,2}F_{1,1}^2F_{1,2} + 3F_{0,2}F_{0,3}F_{1,1}F_{2,0} - 6F_{0,3}F_{1,1}^2F_{2,0} - 3F_{0,2}^2F_{1,2}F_{2,0} + 9F_{0,2}F_{1,1}F_{1,2}F_{2,0} \\
& + 3F_{0,2}F_{0,3}F_{2,0}^2 - 3F_{0,3}F_{1,1}F_{2,0}^2 + 9F_{0,2}F_{1,2}F_{2,0}^2 + 3F_{1,1}F_{1,2}F_{2,0}^2 - F_{0,3}F_{2,0}^3 + 4F_{0,3}F_{1,1}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} \\
& - 6F_{0,2}F_{1,1}F_{1,2}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} - F_{0,2}F_{0,3}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} + 6F_{0,3}F_{1,1}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} \\
& - 9F_{0,2}F_{1,2}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} + 3F_{0,3}F_{2,0}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} + 3F_{1,2}F_{2,0}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} \\
& - 3F_{0,2}^2F_{1,1}F_{2,1} - 9F_{0,2}^2F_{2,0}F_{2,1} - 9F_{0,2}F_{1,1}F_{2,0}F_{2,1} - 6F_{1,1}^2F_{2,0}F_{2,1} + 3F_{0,2}F_{2,0}^2F_{2,1} \\
& + 3F_{0,2}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{2,1} - 9F_{0,2}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{2,1} - 6F_{1,1}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{2,1} + F_{0,2}^3F_{3,0} \\
& + 3F_{0,2}^2F_{1,1}F_{3,0} + 6F_{0,2}F_{1,1}^2F_{3,0} + 4F_{1,1}^3F_{3,0} - 3F_{0,2}^2F_{2,0}F_{3,0} - 3F_{0,2}F_{1,1}F_{2,0}F_{3,0} \\
& + 3F_{0,2}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0} + 6F_{0,2}F_{1,1}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0} + 4F_{1,1}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0} \\
& - F_{0,2}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0}\Big\}
\end{aligned}
$$

- **Show Pick :** [Ruled surfaces $\equiv 0$]

$$
\begin{aligned}
\text{Pick} := \; & 6F_{yyy}F_{xy}F_{xyy}F_{xx}^2 - 9F_{yy}F_{xyy}^2F_{xx}^2 - F_{yyy}^2F_{xx}^3 - 12F_{yyy}F_{xy}^2F_{xx}F_{xxy} + 18F_{yy}F_{xy}F_{xyy}F_{xx}F_{xxy} \\
& + 6F_{yy}F_{yyy}F_{xx}^2F_{xxy} - 9F_{yy}^2F_{xx}F_{xxy}^2 + 8F_{yyy}F_{xy}^3F_{xxx} - 12F_{yy}F_{xy}^2F_{xyy}F_{xxx} - 6F_{yy}F_{yyy}F_{xy}F_{xx}F_{xxx} \\
& + 6F_{yy}^2F_{xyy}F_{xx}F_{xxx} + 6F_{yy}^2F_{xy}F_{xxy}F_{xxx} - F_{yy}^3F_{xxx}^2.
\end{aligned}
$$

Back to $\mathrm{Re}\, w = z\bar{z}$

- **Biholomorphisms :** [Be elementary !]

$$(z,w) \longmapsto \big(f(z,w),\ g(z,w)\big) =: \big(z',\ w'\big).$$

- **One-term groups :**

$$\big(f_t(z,w),\ g_t(z,w)\big)_{t\in\mathbb{C}}.$$

- **Diff $\frac{d}{dt}$:**

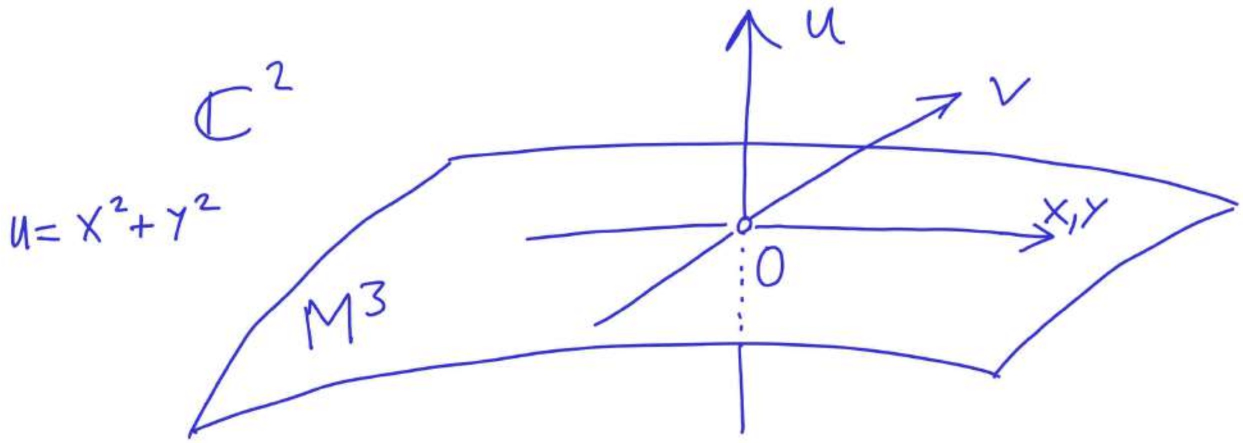
$$L := \frac{d}{dt}\bigg|_{t=0} f_t(z,w)\,\frac{\partial}{\partial z} + \frac{d}{dt}\bigg|_{t=0} g_t(z,w)\,\frac{\partial}{\partial z} =: A(z,w)\,\partial_z + B(z,w)\,\partial_w.$$

- **Holomorphic coefficients :**

$A(z,w)$ and $B(z,w)$.

- **Recall :**

$$(z, w) = (x + i\, y,\ u + i\, v) \in \mathbb{C}^2.$$



- **Graphed real hypersurface :**

[General] $u = F(x, y, v)$

[Quadric] $u = x^2 + y^2.$

- **Proposition.** [Lie, Cartan]

$\exp_{t \in \mathbb{R}} (t\, L)(x, y, u)$ stabilizes S^2 $\iff$ $\big(L + \overline{L}\big)\big|_{S^2}$ tangent to S^2.

$$L + \overline{L} = A(z, w)\, \partial_z + B(z, w)\, \partial_w + \overline{A}(\overline{z}, \overline{w})\, \partial_{\overline{z}} + \overline{B}(\overline{z}, \overline{w})\, \partial_{\overline{w}}.$$

• **Theorem** [Cartan 1932] *Classification of all holomorphically homogeneous hypersurfaces $M^3 \subset \mathbb{C}^2$:* [Nurowski-Tafel 1993]

Si une hypersurface admettant un groupe pseudo-conforme transitif n'est pas localement équivalente à l'hypersphère, elle est globalement équivalente à l'une des hypersurfaces suivantes ou à l'une de leurs variétés de recouvrement: [7 p. 1284]

1° (E) $\frac{y-\overline{y}}{2i} = \left(\frac{x-\overline{x}}{2i}\right)^m$, avec $\frac{x-\overline{x}}{2i} > 0$ $\quad (|m| \geqslant 1,\ m \neq 1, 2)$;

2° (F) $\frac{y-\overline{y}}{2i} = e^{\frac{x-\overline{x}}{y-\overline{y}}}$;

3° (H) $(x-\overline{x})^2 + (y-\overline{y})^2 + 4\, e^{2m \arctan \frac{x-\overline{x}}{y-\overline{y}}} = 0$;

4° (K) $1 + x\overline{x} - y\overline{y} = \mu\, |1 + x^2 - y^2|$, avec $\frac{x(1+\overline{y}) - \overline{x}(1+y)}{i} > 0$ $\quad (\mu > 1)$;

5° (K′) $x\overline{x} + y\overline{y} - 1 = \mu\, |x^2 + y^2 - 1|$, sauf les points réels $(|\mu| < 1,\ \mu \neq 0)$;

6° (L) $x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3 = \mu\, |x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3|$ $\quad (\mu > 1)$.

• **Flat model :** [Easiest !]

$$w + \overline{w} = z\,\overline{z}.$$

• **Weights :**

$$[z] := 1, \qquad [\overline{z}] := 1,$$
$$[w] := 2, \qquad [\overline{w}] := 2.$$

- **Tangency :**

$$0 = L\Big(-w - \overline{w} + z\,\overline{z}\Big)\Big|_{w=-\overline{w}+z\overline{z}}.$$

- **Explicitly :**

$$0 = -B(z,w) - \overline{B}\big(\overline{z},\overline{w}\big) + \overline{z}\,A(z,w) + z\,\overline{A}\big(\overline{z},\overline{w}\big).$$

- **Replace w :** [Identically in $\mathbb{C}\{z,\overline{z},\overline{w}\}$]

$$0 \equiv -B\big(z,\,-\overline{w}+z\overline{z}\big) - \overline{B}\big(\overline{z},\overline{w}\big) + \overline{z}\,A\big(z,\,-\overline{w}+z\overline{z}\big) + z\,\overline{A}\big(\overline{z},\overline{w}\big)$$

$$= \sum_{\mu=0}^{\infty}\ \sum_{i+j+2k=\mu} \underbrace{\text{Coefficient}_{i,j,k}}_{\text{To be computed!}} \cdot z^i\,\overline{z}^j\,\overline{w}^k.$$

- **Weight-by-weight :**

$$0 \equiv -\underbrace{B\big(z,\,-\overline{w}+z\overline{z}\big)}_{\text{weight } \mu} - \underbrace{\overline{B}\big(\overline{z},\overline{w}\big)}_{\text{weight } \mu} + \underbrace{\overline{z}\,A\big(z,\,-\overline{w}+z\overline{z}\big)}_{1+\text{weight } \mu-1} + \underbrace{z\,\overline{A}\big(\overline{z},\overline{w}\big)}_{1+\text{weight } \mu-1}.$$

- **Weight shift :**

$$\ldots \qquad \{A^{\mu-2},\, B^{\mu-1}\}, \qquad \{A^{\mu-1},\, B^{\mu}\}, \qquad \{A^{\mu},\, B^{\mu+1}\}, \qquad \ldots$$

- **Hence :**

$$0 \;=\; \sum_{\mu=0}^{\infty} \Big\{ -\; B^{\mu} - \overline{B}^{\mu} + \overline{z}\, A^{\mu-1} + z\, \overline{A}^{\mu-1} \Big\}.$$

- **Explicitly :**

$$0 \;=\; \sum_{\mu=0}^{\infty} \Big\{ -\sum_{i+2k=\mu} B_{i,k}\, z^{i}\, \big(-\overline{w} + z\overline{z}\big)^{k} - \sum_{i+2k=\mu} \overline{B}_{i,k}\, \overline{z}^{i}\, \overline{w}^{k}$$
$$+\, \overline{z} \sum_{i+2k=\mu-1} A_{i,k}\, z^{i}\, \big(-\overline{w} + z\overline{z}\big)^{k} + z \sum_{i+2k=\mu-1} \overline{A}_{i,k}\, \overline{z}^{i}\, \overline{w}^{k} \Big\}.$$

- **First weights :**

$$
\begin{aligned}
A(z,w) \;=\; & \qquad A_{0,0} + A_{1,0}\,z + A_{2,0}\,z^2 + A_{3,0}\,z^3 + A_{4,0}\,z^4 \\
& + A_{0,1}\,w + A_{1,1}\,zw + A_{2,1}\,z^2w \\
& + A_{0,2}\,w^2 \\
B(z,w) \;=\; & B_{0,0} + B_{1,0}\,z + B_{2,0}\,z^2 + B_{3,0}\,z^3 + B_{4,0}\,z^4 + B_{5,0}\,z^5 \\
& + B_{0,1}\,w + B_{1,1}\,zw + B_{2,1}\,z^2w + B_{3,1}\,z^3w \\
& + B_{0,2}\,w^2 + B_{1,2}\,zw^2
\end{aligned}
$$

$$
AA := z^4 A_{4,0} + w z^2 A_{2,1} + z^3 A_{3,0} + w^2 A_{0,2} + w z A_{1,1} + z^2 A_{2,0} + w A_{0,1} + z A_{1,0} + A_{0,0}
$$

$$
BB := z^5 B_{5,0} + w z^3 B_{3,1} + z^4 B_{4,0} + w^2 z B_{1,2} + w z^2 B_{2,1} + z^3 B_{3,0} + w^2 B_{0,2} + w z B_{1,1} + z^2 B_{2,0} + w B_{0,1} + z B_{1,0} + B_{0,0}
$$

$$AAb := \overline{A_{0,0}} + \overline{A_{1,0}}\,zb + \overline{A_{2,0}}\,zb^2 + \overline{A_{0,1}}\,wb + \overline{A_{3,0}}\,zb^3 + \overline{A_{1,1}}\,zb\,wb + \overline{A_{4,0}}\,zb^4 + \overline{A_{2,1}}\,zb^2\,wb + \overline{A_{0,2}}\,wb^2$$

$$BBb := \overline{B_{0,0}} + \overline{B_{1,0}}\,zb + \overline{B_{2,0}}\,zb^2 + \overline{B_{0,1}}\,wb + \overline{B_{3,0}}\,zb^3 + \overline{B_{1,1}}\,zb\,wb + \overline{B_{4,0}}\,zb^4 + \overline{B_{2,1}}\,zb^2\,wb + \overline{B_{0,2}}\,wb^2 + \overline{B_{5,0}}\,zb^5 + \overline{B_{3,1}}\,zb^3\,wb + \overline{B_{1,2}}\,zb\,wb^2$$

- **Infinitesimal symmetry :**

$$LL := \left(A_{4,0}\,z^4 + A_{2,1}\,z^2\,w + A_{3,0}\,z^3 + A_{0,2}\,w^2 + A_{1,1}\,z\,w + A_{2,0}\,z^2 + A_{0,1}\,w + A_{1,0}\,z + A_{0,0}\right)\partial_z + \left(B_{5,0}\,z^5 + B_{3,1}\,z^3\,w + B_{4,0}\,z^4 + B_{1,2}\,z\,w^2 + B_{2,1}\,z^2\,w + B_{3,0}\,z^3 + B_{0,2}\,w^2 + B_{1,1}\,z\,w + B_{2,0}\,z^2 + B_{0,1}\,w + B_{1,0}\,z + B_{0,0}\right)\partial_w + \left(\overline{A_{0,0}} + \overline{A_{1,0}}\,zb + \overline{A_{2,0}}\,zb^2 + \overline{A_{0,1}}\,wb + \overline{A_{3,0}}\,zb^3 + \overline{A_{1,1}}\,zb\,wb + \overline{A_{4,0}}\,zb^4 + \overline{A_{2,1}}\,zb^2\,wb + \overline{A_{0,2}}\,wb^2\right)\partial_{zb} + \left(\overline{B_{0,0}} + \overline{B_{1,0}}\,zb + \overline{B_{2,0}}\,zb^2 + \overline{B_{0,1}}\,wb + \overline{B_{3,0}}\,zb^3 + \overline{B_{1,1}}\,zb\,wb + \overline{B_{4,0}}\,zb^4 + \overline{B_{2,1}}\,zb^2\,wb + \overline{B_{0,2}}\,wb^2 + \overline{B_{5,0}}\,zb^5 + \overline{B_{3,1}}\,zb^3\,wb + \overline{B_{1,2}}\,zb\,wb^2\right)\partial_{wb}$$

- **Equation :**

$$0 = \mathsf{eqL}.$$

$$eqL := (A_{4,0} - B_{3,1})\, zb\, z^4 + B_{3,1}\, wb\, z^3 + (A_{2,1} - B_{1,2})\, zb^2 z^3 + A_{0,2}\, z^2 zb^3 + (\overline{A_{0,2}} - B_{1,2})\, wb^2 z + \overline{A_{4,0}}\, zb^4 z + (A_{3,0} - B_{2,1})\, zb\, z^3 + B_{2,1}\, wb\, z^2 + (A_{1,1} - B_{0,2})\, zb^2 z^2 + \overline{A_{3,0}}\, zb^3 z + (-\overline{B_{0,2}} - B_{0,2})\, wb^2 - \overline{B_{0,0}} + (\overline{A_{1,0}} + A_{1,0} - B_{0,1})\, zb\, z + (-\overline{B_{0,1}} + B_{0,1})\, wb + (A_{2,0} - B_{1,1})\, zb\, z^2 + (\overline{A_{0,1}} + B_{1,1})\, wb\, z + (\overline{A_{2,0}} + A_{0,1})\, zb^2 z - \overline{B_{3,1}}\, zb^3 wb + (\overline{A_{0,0}} - B_{1,0})\, z + (-\overline{B_{1,0}} + A_{0,0})\, zb - \overline{B_{4,0}}\, zb^4 - \overline{B_{2,1}}\, zb^2 wb - \overline{B_{5,0}}\, zb^5 - \overline{B_{3,0}}\, zb^3 - \overline{B_{2,0}}\, zb^2 - B_{0,0} + (-\overline{B_{1,1}} - A_{0,1})\, wb\, zb + (\overline{A_{1,1}} - A_{1,1} + 2\, B_{0,2})\, wb\, zb\, z + (-A_{2,1} + 2\, B_{1,2})\, wb\, z^2 zb + (\overline{A_{2,1}} - 2\, A_{0,2})\, wb\, zb^2 z + (-\overline{B_{1,2}} + A_{0,2})\, wb^2 zb - B_{2,0}\, z^2 - B_{3,0}\, z^3 - B_{5,0}\, z^5 - B_{4,0}\, z^4$$

$$eqL0 := -\overline{B_{0,0}} - B_{0,0}$$

$$eqL1 := (\overline{A_{0,0}} - B_{1,0})\, z + (-\overline{B_{1,0}} + A_{0,0})\, zb$$

$$eqL2 := -B_{2,0}\, z^2 + (\overline{A_{1,0}} + A_{1,0} - B_{0,1})\, zb\, z + (-\overline{B_{0,1}} + B_{0,1})\, wb - \overline{B_{2,0}}\, zb^2$$

$$eqL3 := -B_{3,0}\, z^3 + (A_{2,0} - B_{1,1})\, zb\, z^2 + (\overline{A_{0,1}} + B_{1,1})\, wb\, z + (\overline{A_{2,0}} + A_{0,1})\, zb^2\, z + (-\overline{B_{1,1}} - A_{0,1})\, wb\, zb - \overline{B_{3,0}}\, zb^3$$

$$eqL4 := -B_{4,0}\, z^4 + (A_{3,0} - B_{2,1})\, zb\, z^3 + B_{2,1}\, wb\, z^2 + (A_{1,1} - B_{0,2})\, zb^2\, z^2 + (\overline{A_{1,1}} - A_{1,1} + 2\, B_{0,2})\, wb\, zb\, z + \overline{A_{3,0}}\, zb^3\, z + (-\overline{B_{0,2}} - B_{0,2})\, wb^2 - \overline{B_{2,1}}\, zb^2\, wb - \overline{B_{4,0}}\, zb^4$$

$$eqL5 := -B_{5,0}\, z^5 + (A_{4,0} - B_{3,1})\, zb\, z^4 + B_{3,1}\, wb\, z^3 + (A_{2,1} - B_{1,2})\, zb^2\, z^3 + A_{0,2}\, z^2\, zb^3 + (-A_{2,1} + 2\, B_{1,2})\, wb\, z^2\, zb + (\overline{A_{0,2}} - B_{1,2})\, wb^2\, z + (\overline{A_{2,1}} - 2\, A_{0,2})\, wb\, zb^2\, z + \overline{A_{4,0}}\, zb^4\, z + (-\overline{B_{1,2}} + A_{0,2})\, wb^2\, zb - \overline{B_{3,1}}\, zb^3\, wb - \overline{B_{5,0}}\, zb^5$$

- **Order 0 :**

$$B_{0,0} := \sqrt{-1}\, ImB_{0,0}.$$

- **Order 1 :**

$$B_{1,0} := \overline{A_{0,0}}.$$

- **Order 2 :**

$$\begin{aligned} B_{2,0} &:= 0, \\ B_{0,1} &:= ReB_{0,1}, \\ A_{1,0} &:= ReA_{1,0} + \sqrt{-1}\, ImA_{1,0}, \\ ReB_{0,1} &:= 2\, ReA_{1,0}. \end{aligned}$$

- **Order 3 :**

$$\begin{aligned} B_{3,0} &:= 0, \\ B_{1,1} &:= A_{2,0}, \\ A_{0,1} &:= -\overline{A_{2,0}}. \end{aligned}$$

- **Order 4 :**

$$\begin{aligned} B_{4,0} &:= 0, \\ B_{2,1} &:= A_{3,0}, \\ A_{3,0} &:= 0, \\ B_{0,2} &:= ImB_{0,2}. \end{aligned}$$

- **Proposition.** *In all weight $\mu \geqslant 5$:*

$$0 \equiv A^{\mu-1} \equiv B^{\mu}. \qquad \square$$

- **After resolution :**

$$LL := \left(\mathrm{I}\, ImA20\, z^2 + \mathrm{I}\, ImB02\, z\, w + \mathrm{I}\, ImA10\, z + \mathrm{I}\, ImA20\, w + ReA20\, z^2 + \mathrm{I}\, ImA00 + ReA10\, z - ReA20\, w + ReA00\right) \partial_z - \left(-\mathrm{I}\, ImA20\, w\, z - \mathrm{I}\, ImB02\, w^2 + \mathrm{I}\, ImA00\, z - ReA20\, w\, z - \mathrm{I}\, ImB00 - ReA00\, z - 2\, ReA10\, w\right) \partial_w$$

- **8 generators :**

$$e_1 := \partial_z + z\,\partial_w$$

$$e_2 := \mathrm{I}\,\partial_z - \mathrm{I}\,z\,\partial_w$$

$$e_3 := 0\,\partial_z + \mathrm{I}\,\partial_w$$

$$e_4 := z\,\partial_z + 2\,w\,\partial_w$$

$$e_5 := \mathrm{I}\,z\,\partial_z + 0\,\partial_w$$

$$e_6 := -\left(-z^2 + w\right)\partial_z + w\,z\,\partial_w$$

$$e_7 := \left(\mathrm{I}\,z^2 + \mathrm{I}\,w\right)\partial_z + \mathrm{I}\,w\,z\,\partial_w$$

$$e_8 := \mathrm{I}\,w\,z\,\partial_z + \mathrm{I}\,w^2\,\partial_w$$

- **Lie structure :** $[\mathfrak{su}(2,1)]$

Algebre_Lie := [*e1*, *e2*] = − 2 *e3*, [*e1*, *e3*] = 0, [*e1*, *e4*] = *e1*, [*e1*, *e5*] = *e2*, [*e1*, *e6*] = *e4*, [*e1*, *e7*] = 3 *e5*, [*e1*, *e8*] = *e7*, [*e2*, *e3*] = 0, [*e2*, *e4*] = *e2*, [*e2*, *e5*] = − *e1*, [*e2*, *e6*] = 3 *e5*, [*e2*, *e7*] = − *e4*, [*e2*, *e8*] = *e6*, [*e3*, *e4*] = 2 *e3*, [*e3*, *e5*] = 0, [*e3*, *e6*] = − *e2*, [*e3*, *e7*] = − *e1*, [*e3*, *e8*] = − *e4*, [*e4*, *e5*] = 0, [*e4*, *e6*] = *e6*, [*e4*, *e7*] = *e7*, [*e4*, *e8*] = 2 *e8*, [*e5*, *e6*] = *e7*, [*e5*, *e7*] = − *e6*, [*e5*, *e8*] = 0, [*e6*, *e7*] = − 2 *e8*, [*e6*, *e8*] = 0, [*e7*, *e8*] = 0

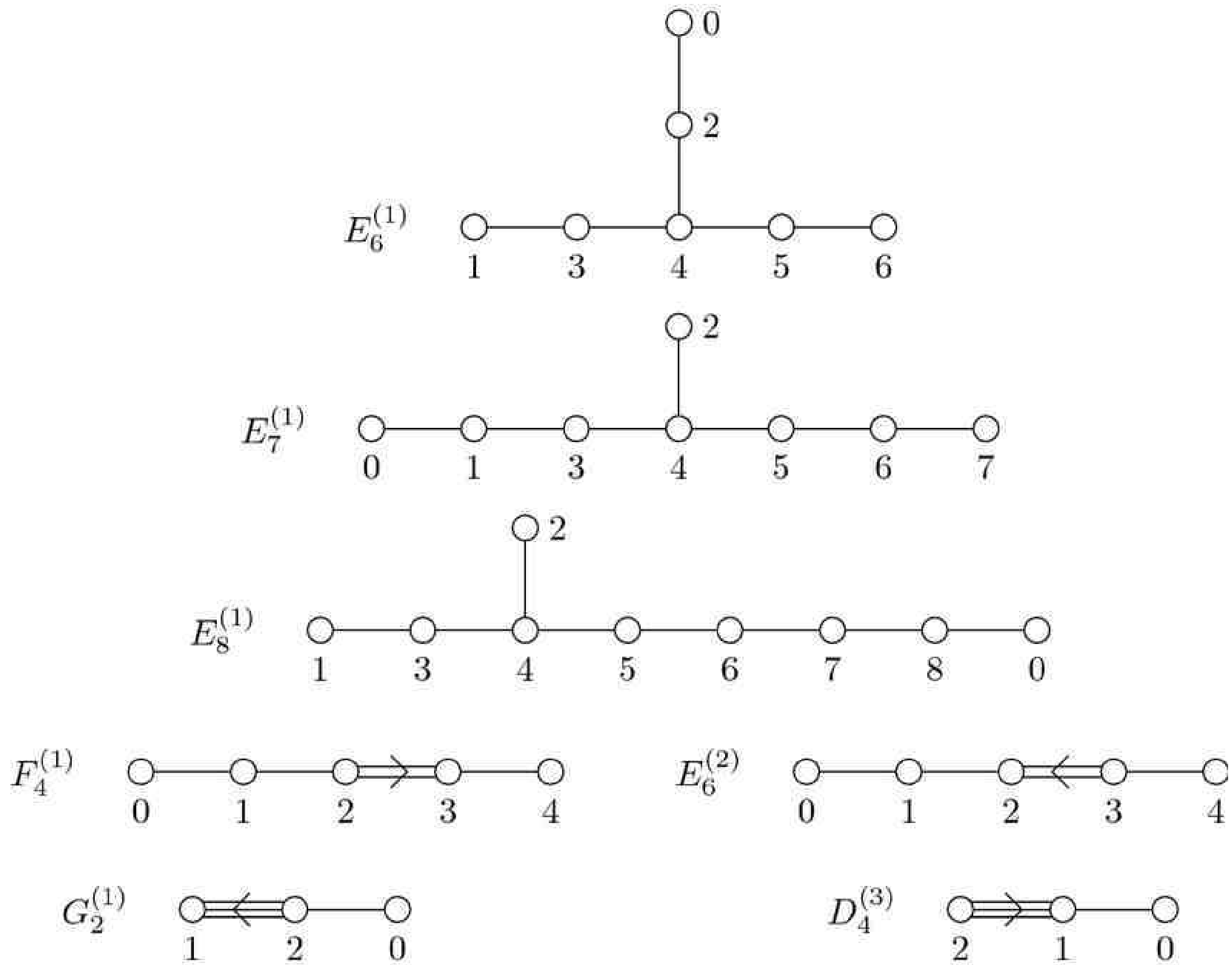
semi_simple := *true*

Interlude

- **Tangency :** [Lecture 1]
  - ☐ Give a geometric model.
  - ☐ Find its infinitesimal symmetries L.

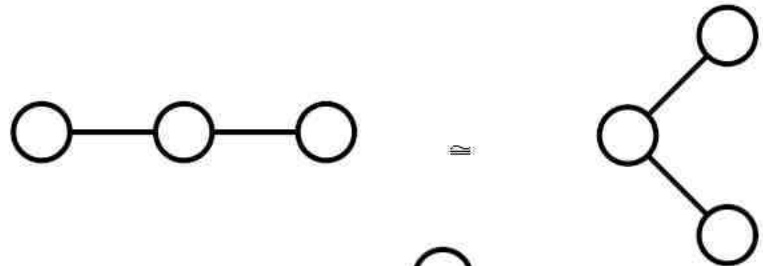
- **Power series method of equivalence :** [Lecture 2]
  - ☐ Proceed order by order.
  - ☐ Normalize Taylor coefficients.
  - ☐ Linearize the action.
  - ☐ Apply Linear Representation Theory order by order.
  - ☐ Treat equivalence modulo discrete groups.

Dynkin diagrams

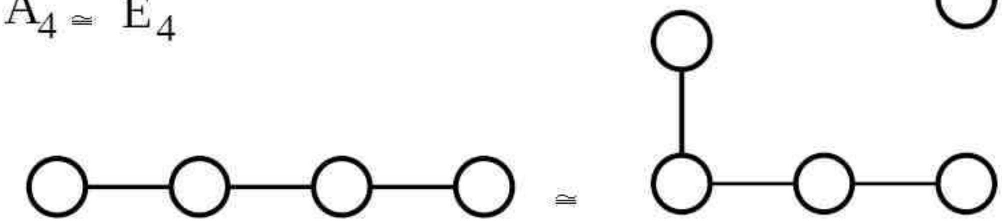


$B_2 \cong C_2$

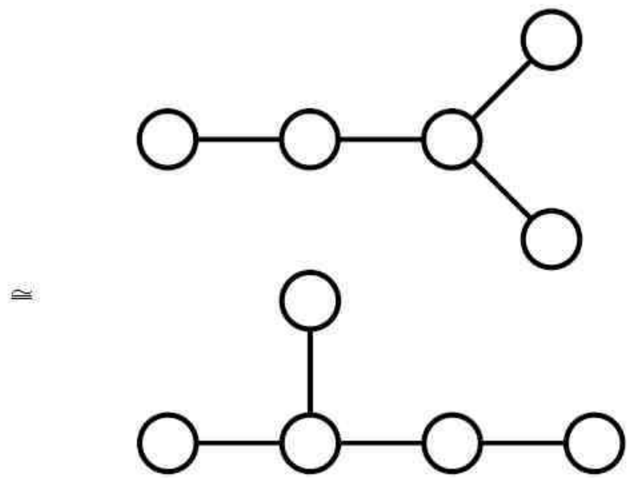
$A_3 \cong D_3$



$A_4 \cong E_4$



$D_5 \cong E_5$

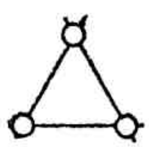
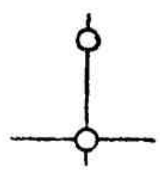


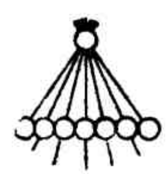
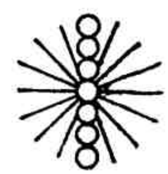
Subgroups of the Projective Group

Im Ganzen giebt es also innerhalb der allgemeinen projectiven Gruppe der Ebene die folgenden verschiedenen Typen von infinitesimalen Transformationen:

$$(8) \quad \begin{cases} \boxed{xp + cyq \quad (c \neq 0, 1)} \\ \boxed{p + yq} \quad \boxed{p + xq} \quad \boxed{yq} \quad \boxed{q}\,. \end{cases}$$

Die Punkte und Geraden, die bei diesen infinitesimalen Transformationen in Ruhe bleiben, bilden der Reihe nach die folgenden Figuren:

Dritter Typus.

Bei der eingliedrigen Gruppe des *dritten* Typus $p + xq$ erhalten wir als die Bahncurven die Integralcurven der Differentialgleichung:

$$\frac{dx}{1} = \frac{dy}{x},$$

d. h. die Curven zweiten Grades

$$y - \tfrac{1}{2}x^2 = \text{Const.}$$

Die elementare analytische Theorie der *Curven zweiten Grades oder*

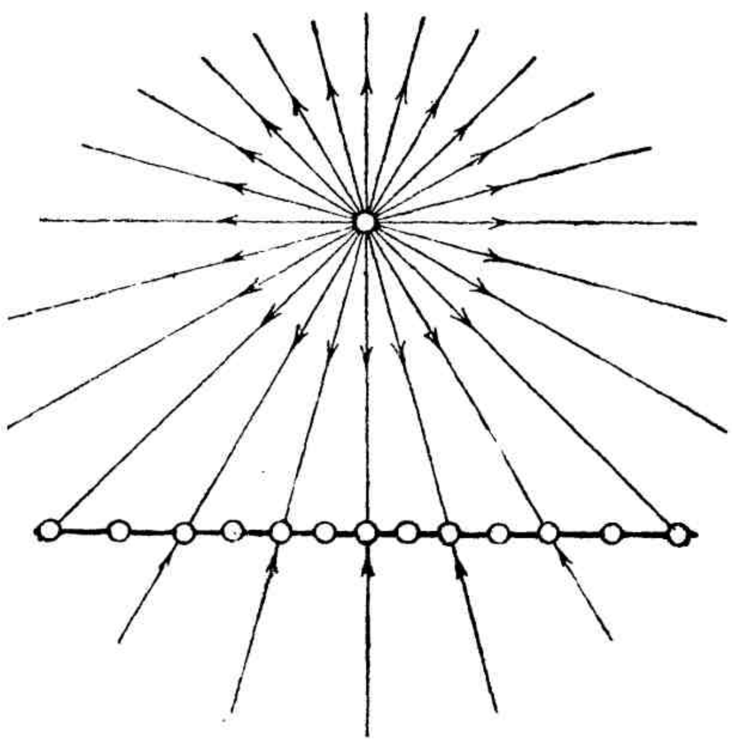


Fig. 14.

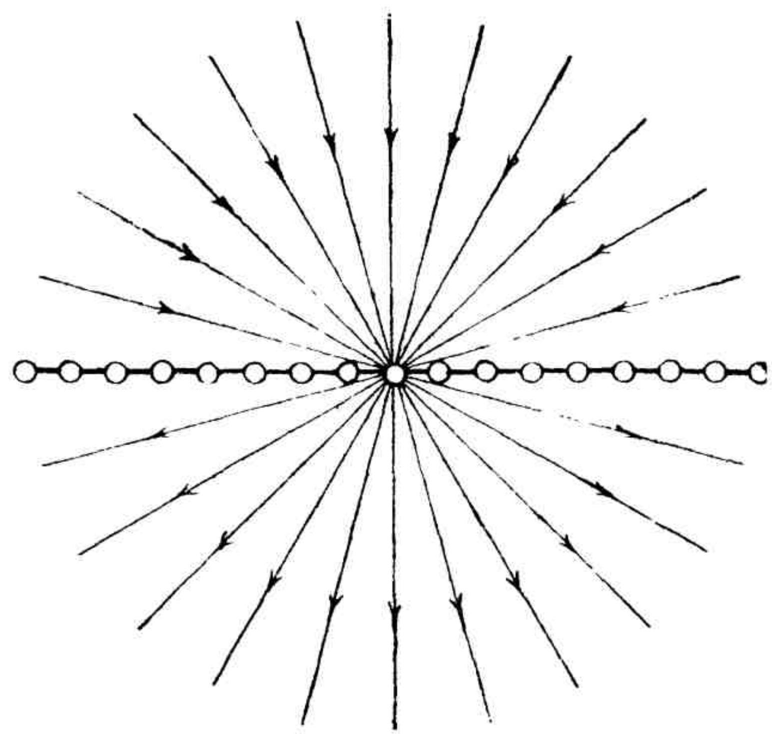


Fig. 15.

Verlauf der Bahncurven. Wählt man die beiden invarianten Punkte als die unendlich fernen Punkte der Axen und die eine invariante Gerade als x-Axe, so hat Uf die Form $p + yq$ und die Bahncurven werden die Integralcurven von

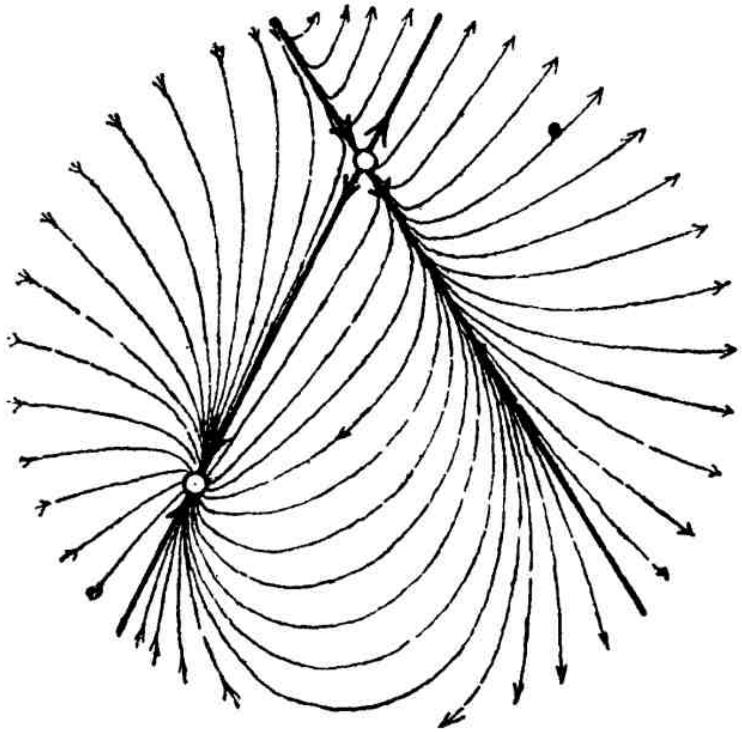


Fig. 22.

$$\frac{dx}{1} = \frac{dy}{y},$$

d. h. die *transcendenten* Curven

$$y = \text{Const.}\, e^x.$$

Dieselben gehen alle durch die beiden invarianten Punkte, was daraus folgt, dass

$$\frac{y}{x} = \frac{y}{\lg y + \text{Const.}}$$

Exceptional CR Geometries

Denson Hill
Paweł Nurowski
Zhaohu Nie

From Paweł Nurowski's notes :

In a German version of his Ph.D. thesis, Élie Cartan gives a realization of the simple exceptional Lie group F_4 as a symmetry group of a certain rank 8 vector distribution in dimension 15.

[*Über die einfachen Transformationsgruppen*, Leipz. Ber. 1893, 395–420.]

Sigurdur Helgason recalls this fact.

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INVARIANT DIFFERENTIAL EQUATIONS ON HOMOGENEOUS MANIFOLDS

BY SIGURDUR HELGASON[1]

1. Historical origins of Lie group theory. Nowadays when Lie groups enter

In this remarkable work, Killing finds all possibilities for the matrix (a_{ij}) and writes down the corresponding roots $\omega(X)$ (cf. [**5**, II, §15]). Thus he arrives at the statement that apart from the classical simple Lie algebras

$$A_l\ (l \geqslant 1), \quad B_l\ (l \geqslant 2), \quad C_l\ (l \geqslant 3), \quad D_l\ (l \geqslant 3)$$

(known from Lie's work), there are only six more, of ranks and dimension, respectively,

$$l = 2, 4, 4, 6, 7, 8,$$
$$r = 14, 52, 52, 78, 133, 248.$$

These exceptional Lie algebras are denoted G_2, E_4, F_4, E_6, E_7, E_8, respectively. Killing denoted G_2 by (IIC); he observed that $A_3 = D_3$, but did not notice that $E_4 = F_4$, although, as Cartan remarked, this is immediate from his

In his thesis [**1b**], É. Cartan gave a complete proof of the classification results stated by Killing; in outline his method follows Killing's program. He determined the matrices (a_{ij}), the roots $\omega(X)$ and a basis for each of the exceptional Lie algebras with respect to which the structural constants have a simple and symmetric form [**1b**, §§18–20] whereby the Jacobi identity (7) is (presumably) simple to verify.[4] But he was also interested in realizing the exceptional Lie groups by transformations, like e.g. the classical algebra C_l is the Lie algebra of the linear group leaving invariant the Pfaffian form

$$x_1 dy_1 - y_1 dx_1 + \cdots + x_l dy_l - y_l dx_l.$$

Killing had been led to expect that G_2 could be realized as a transformation group in $\mathbf{R}^5$, but not in a lower-dimensional space. Engel and Cartan showed that it can be realized as the stability group of the system

$$\begin{aligned} dx_3 + x_1 dx_2 - x_2 dx_1 &= 0, \\ dx_4 + x_3 dx_1 - x_1 dx_3 &= 0, \\ dx_5 + x_2 dx_3 - x_3 dx_2 &= 0, \end{aligned}$$

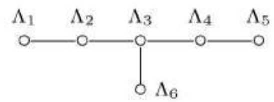
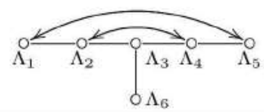
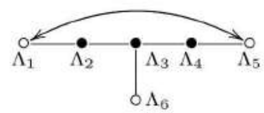
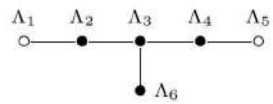
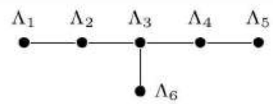
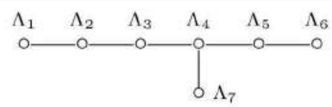
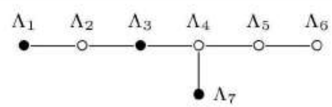
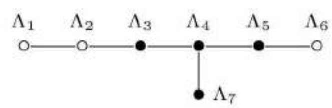
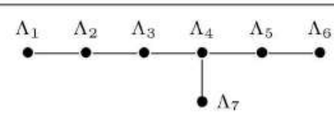
in $\mathbf{R}^5$ (Engel [**3a**], Cartan [**1b**, p. 281], Lie and Engel [**9**, vol. 3, p. 764]).

Cartan represented F_4 similarly by the Pfaffian system in $\mathbf{R}^{15}$ given by

$$(14) \qquad dz = \sum_{1}^{4} y_i dx_i, \qquad dx_{ij} = x_i dx_j - x_j dx_i + y_h dy_k - y_k dy_h,$$

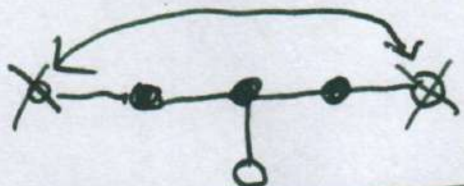
where $z, x_i, y_j, x_{ij} = -x_{ji}$ $(i \neq j, i, j = 1, 2, 3, 4)$ are coordinates in $\mathbf{R}^{15}$ and in (14) i, j, h, k is an even permutation [**1a**, p. 418]. Similar results for E_6 in $\mathbf{R}^{16}$, E_7 in $\mathbf{R}^{27}$ and E_8 in $\mathbf{R}^{29}$ as contact transformations are indicated in [**1a**]. Unfortunately, detailed proofs of these remarkable representations of the exceptional groups do not seem to be available.

Satake diagrams

Real form	Satake diagram with a weight	s	ν	Index
EI		e	$\neq e$	$+1$
EII		$\neq e$	$\neq e$	$+1$
EIII		$\neq e$	$\neq e$	$+1$
EIV		e	$\neq e$	$+1$
compact form of E_6		$\neq e$	$\neq e$	$+1$
EV		e	e	$+1$
EVI		e	e	$(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$
EVII		e	e	$+1$
compact form of E_7		e	e	$(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$

From Paweł Nurowski's notes

(III) CR manifolds of complex dimension 8 and real codimension 8 with E_6 symmetry

(III 1) The  case.

Somehow an analog of Cartan's Theorem 1 for a rank 16 distributions in $\mathbb{R}^{24}$ can be formulated as follows.

Let (x^i) $i=1,2,\dots 24$, be coordinates in $\mathbb{R}^{24}$.

Consider a distribution $\mathcal{D}$ on $\mathbb{R}^{24}$ being the annihilator of eight 1-forms $[\lambda^1, \lambda^2, \dots \lambda^8]$ such that:

$$\lambda^1 = dx^1 + x^9 dx^{24} + x^{10} dx^{20} + x^{11} dx^{23} - x^{12} dx^{18} + x^{13} dx^{22} - x^{14} dx^{21} - x^{15} dx^{19} - x^{16} dx^{17}$$

$$\lambda^2 = dx^2 - x^9 dx^{20} + x^{10} dx^{24} + x^{11} dx^{22} + x^{12} dx^{17} - x^{13} dx^{23} - x^{14} dx^{19} + x^{15} dx^{21} - x^{16} dx^{18}$$

$$\lambda^3 = dx^3 - x^9 dx^{23} - x^{10} dx^{22} + x^{11} dx^{24} - x^{12} dx^{21} + x^{13} dx^{20} + x^{14} dx^{18} + x^{15} dx^{17} - x^{16} dx^{19} \qquad \text{(HNN1)}$$

$$\lambda^4 = dx^4 - x^9 dx^{10} - x^{11} dx^{13} - x^{12} dx^{16} - x^{14} dx^{15} - x^{17} dx^{18} - x^{19} dx^{21} - x^{20} dx^{24} - x^{22} dx^{23}$$

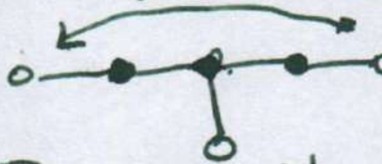
$$\lambda^5 = dx^5 - x^9 dx^{22} + x^{10} dx^{23} - x^{11} dx^{20} + x^{12} dx^{19} + x^{13} dx^{24} + x^{14} dx^{17} - x^{15} dx^{18} - x^{16} dx^{21}$$

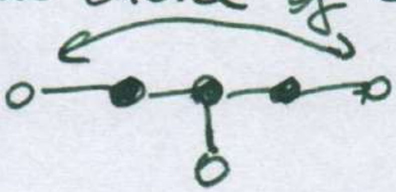
$$\lambda^6 = dx^6 - x^9 dx^{13} - x^{10} dx^{11} + x^{12} dx^{15} - x^{14} dx^{16} - x^{17} dx^{21} - x^{18} dx^{19} + x^{20} dx^{23} - x^{22} dx^{24}$$

$$\lambda^7 = dx^7 - x^9 dx^{11} + x^{10} dx^{13} - x^{12} dx^{14} - x^{15} dx^{16} - x^{17} dx^{19} + x^{18} dx^{21} - x^{20} dx^{22} - x^{23} dx^{24}$$

$$\lambda^8 = dx^8 - x^9 dx^{17} - x^{10} dx^{18} - x^{11} dx^{19} + x^{12} dx^{20} - x^{13} dx^{21} + x^{14} dx^{22} + x^{15} dx^{23} + x^{16} dx^{24}$$

Theorem 2

Distribution $\mathcal{D}$ annihilating forms $[\lambda^1, \dots \lambda^8]$ above has the real form  of the simple exceptional group E_6 as its local group of symmetries.

Such a distribution on $\mathbb{R}^{24}$ is locally isomorphic to the flat model of parabolic geometry  of type (E_6, P) where P is a parabolic subgroup in E_6 corresponding to the choice of extreme roots in the Satake diagram .

Structure Constants of E_6 II

	100000	010000	001000	000100	000010	000001	101000	010100	001100	000110	000011	101100	011100	010110	001110	000111	111100	101110	011110	010111	001111	111110	101111	011210	011111	111210	111111	011211	112210	111211	011221	112211	111221	112221	112321	122321
100000	0	0	+	0	0	0	0	0	+	0	0	0	+	0	+	0	0	0	+	0	+	0	0	+	+	0	0	+	0	0	+	0	0	0	0	0
010000	0	0	0	+	0	0	0	0	+	+	0	+	0	0	+	+	0	+	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0	0	+	0
001000	−	0	0	+	0	0	0	+	0	+	0	0	0	+	0	+	0	0	0	+	0	0	0	0	0	+	0	0	0	+	0	0	+	0	0	0
000100	0	−	−	0	+	0	−	0	0	0	+	0	0	0	0	0	0	0	+	0	0	+	0	0	+	0	+	0	0	0	0	0	0	+	0	0
000010	0	0	0	−	0	+	0	−	−	0	0	−	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	0	0	0
000001	0	0	0	0	−	0	0	0	0	−	0	0	0	−	−	0	0	−	−	0	0	−	0	−	0	−	0	0	−	0	0	0	0	0	0	0
101000	0	0	0	+	0	0	0	+	0	+	0	0	0	+	0	+	0	0	0	+	0	0	0	−	0	0	0	−	0	0	−	0	0	0	0	0
010100	0	0	−	0	+	0	−	0	0	0	+	0	0	0	−	0	0	−	0	0	−	0	−	0	0	0	0	0	0	0	0	0	0	+	0	0
001100	−	−	0	0	+	0	0	0	0	0	+	0	0	−	0	0	0	0	0	−	0	+	0	0	0	0	+	0	0	0	0	0	−	0	0	0
000110	0	−	−	0	0	+	−	0	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	−	0	−	0	0	0	0	+	0	0	0	0
000011	0	0	0	−	0	0	0	−	−	0	0	−	−	0	0	0	−	0	0	0	0	0	0	−	0	−	0	0	−	0	0	0	0	0	0	0
101100	0	−	0	0	+	0	0	0	0	0	+	0	0	−	0	0	0	0	−	−	0	0	0	0	−	0	0	0	0	0	+	0	0	0	0	0
011100	−	0	0	0	+	0	0	0	0	+	+	0	0	0	0	+	0	−	0	0	0	0	−	0	0	0	0	0	0	0	0	0	−	0	0	0
010110	0	0	−	0	0	+	−	0	+	0	0	+	0	0	0	0	0	0	0	0	+	0	+	0	0	0	0	0	0	0	0	+	0	0	0	0
001110	−	−	0	0	0	+	0	+	0	0	0	0	0	0	0	0	−	0	0	+	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	0
000111	0	−	−	0	0	0	−	0	0	0	0	0	−	0	0	0	−	0	+	0	0	+	0	0	0	0	0	0	−	0	0	0	0	0	0	0
111100	0	0	0	0	+	0	0	0	0	+	+	0	0	0	+	+	0	0	0	0	+	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0
101110	0	−	0	0	0	+	0	+	0	0	0	0	+	0	0	0	0	0	0	+	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0
011110	−	0	0	−	0	+	0	0	0	0	0	+	0	0	0	−	0	0	0	0	0	0	+	0	0	0	0	0	0	−	0	0	0	0	0	0
010111	0	0	−	0	0	0	−	0	+	0	0	+	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0
001111	−	−	0	0	0	0	0	+	0	0	0	0	0	−	0	0	−	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0
111110	0	0	0	−	0	+	0	0	−	0	0	0	0	0	0	−	0	0	0	0	−	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0
101111	0	−	0	0	0	0	0	+	0	0	0	0	+	−	0	0	0	0	−	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0
011210	−	0	0	0	0	+	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0
011111	−	0	0	−	0	0	0	0	0	+	0	+	0	0	0	0	0	−	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0
111210	0	0	−	0	0	+	0	0	0	0	+	0	0	0	0	0	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0
111111	0	0	0	−	0	0	0	0	−	+	0	0	0	0	+	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0
011211	−	0	0	0	−	0	+	0	0	0	0	0	0	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112210	0	0	0	0	0	+	0	0	0	0	+	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
111211	0	0	−	0	−	0	0	0	0	0	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
011221	−	0	0	0	0	0	+	0	0	0	0	−	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112211	0	0	0	0	−	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
111221	0	0	−	0	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112221	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112321	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
122321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

• **In $\mathbb{C}^{8+8}$, consider :** [E_6 II]

$$
\begin{aligned}
\mathrm{Re}\, w_1 &= \mathrm{Re}\,\big(z_1\,\overline{z}_4 + z_2\,\overline{z}_3\big),\\
\mathrm{Re}\, w_2 &= \mathrm{Re}\,\big(z_1\,\overline{z}_6 + z_2\,\overline{z}_5\big),\\
\mathrm{Im}\, w_3 &= \mathrm{Im}\,\big(z_1\,\overline{z}_7 + z_5\,\overline{z}_3\big),\\
\mathrm{Im}\, w_4 &= \mathrm{Im}\,\big(z_2\,\overline{z}_7 + z_3\,\overline{z}_6 - z_5\,\overline{z}_4 - z_8\,\overline{z}_1\big),\\
\mathrm{Re}\, w_5 &= \mathrm{Re}\,\big(z_2\,\overline{z}_7 + z_3\,\overline{z}_6 - z_5\,\overline{z}_4 - z_8\,\overline{z}_1\big),\\
\mathrm{Im}\, w_6 &= \mathrm{Im}\,\big(z_2\,\overline{z}_8 + z_6\,\overline{z}_4\big),\\
\mathrm{Re}\, w_7 &= \mathrm{Re}\,\big(z_3\,\overline{z}_8 + z_4\,\overline{z}_7\big),\\
\mathrm{Re}\, w_8 &= \mathrm{Re}\,\big(z_5\,\overline{z}_8 + z_6\,\overline{z}_7\big).
\end{aligned}
$$

• **In $\mathbb{C}^{8+8}$, consider :** [E_6 III]

$$
\begin{aligned}
\mathrm{Re}\, w_1 &= \mathrm{Re}\,\big(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\big),\\
\mathrm{Re}\, w_2 &= \mathrm{Re}\,\big(|z_5|^2 + |z_6|^2 + |z_7|^2 + |z_8|^2\big),\\
\mathrm{Im}\, w_3 &= \mathrm{Im}\,\big(z_1\,\overline{z}_7 + z_2\,\overline{z}_8 + z_5\,\overline{z}_3 + z_6\,\overline{z}_4\big),\\
\mathrm{Re}\, w_4 &= \mathrm{Re}\,\big(z_1\,\overline{z}_7 + z_2\,\overline{z}_8 + z_5\,\overline{z}_3 + z_6\,\overline{z}_4\big),\\
\mathrm{Im}\, w_5 &= \mathrm{Im}\,\big(z_1\,\overline{z}_6 - z_3\,\overline{z}_8 + z_5\,\overline{z}_2 - z_7\,\overline{z}_4\big),\\
\mathrm{Re}\, w_6 &= \mathrm{Re}\,\big(z_1\,\overline{z}_6 - z_3\,\overline{z}_8 + z_5\,\overline{z}_2 - z_7\,\overline{z}_4\big),\\
\mathrm{Im}\, w_7 &= \mathrm{Im}\,\big(z_2\,\overline{z}_6 + z_3\,\overline{z}_7 - z_5\,\overline{z}_1 - z_8\,\overline{z}_4\big),\\
\mathrm{Re}\, w_8 &= \mathrm{Re}\,\big(z_2\,\overline{z}_6 + z_3\,\overline{z}_7 - z_5\,\overline{z}_1 - z_8\,\overline{z}_4\big).
\end{aligned}
$$

- **What infinitesimal symmetries look like :**

$$
\begin{aligned}
L^2_{w_4 w_4} := \ & (2\,w_1\,z_5 - 2\,w_2\,z_3 - 2\,w_3\,z_2 - w_4\,z_1 + w_5\,z_1)\,\partial_{z_1} \\
& + (2\,w_1\,z_6 - 2\,w_2\,z_4 - w_4\,z_2 - w_5\,z_2 - 2\,w_6\,z_1)\,\partial_{z_2} \\
& + (2\,w_1\,z_7 + 2\,w_3\,z_4 - w_4\,z_3 - w_5\,z_3 - 2\,w_7\,z_1)\,\partial_{z_3} \\
& + (2\,w_1\,z_8 - w_4\,z_4 + w_5\,z_4 + 2\,w_6\,z_3 - 2\,w_7\,z_2)\,\partial_{z_4} \\
& + (2\,w_2\,z_7 + 2\,w_3\,z_6 - w_4\,z_5 - w_5\,z_5 - 2\,w_8\,z_1)\,\partial_{z_5} \\
& + (2\,w_2\,z_8 - w_4\,z_6 + w_5\,z_6 + 2\,w_6\,z_5 - 2\,w_8\,z_2)\,\partial_{z_6} \\
& - (2\,w_3\,z_8 + w_4\,z_7 - w_5\,z_7 - 2\,w_7\,z_5 + 2\,w_8\,z_3)\,\partial_{z_7} \\
& - (w_4\,z_8 + w_5\,z_8 + 2\,w_6\,z_7 - 2\,w_7\,z_6 + 2\,w_8\,z_4)\,\partial_{z_8} \\
& - 2\,w_1\,w_4\,\partial_{w_1} - 2\,w_2\,w_4\,\partial_{w_2} - 2\,w_3\,w_4\,\partial_{w_3} \\
& + (4\,w_1\,w_8 - 4\,w_2\,w_7 - 4\,w_3\,w_6 - w_4^2 - w_5^2)\,\partial_{w_4} \\
& - 2\,w_4\,w_5\,\partial_{w_5} - 2\,w_4\,w_6\,\partial_{w_6} - 2\,w_4\,w_7\,\partial_{w_7} - 2\,w_4\,w_8\,\partial_{w_8}.
\end{aligned}
$$

$\mathfrak{su}(p,q)$ **CR Models :**

Denson Hill
Paweł Nurowski
Zhaohu Nie

- **Equations for** $1 \leqslant a < c \leqslant \ell$ **:**

$$\mathsf{Im}\, w_{ac} = \mathsf{Im} \left\{ \begin{array}{c} z_{a,1}\, \overline{z}_{c,1} + \cdots + z_{a,n}\, \overline{z}_{n,c} \\ +u_{a,1}\, \overline{v}_{c,1} + \cdots + u_{a,m}\, \overline{v}_{c,m} \\ +v_{a,1}\, \overline{u}_{c,1} + \cdots + v_{a,m}\, \overline{u}_{c,m} \end{array} \right\},$$

and for $1 \leqslant c \leqslant a \leqslant \ell$:

$$\mathsf{Re}\, w_{ac} = \mathsf{Re} \left\{ \begin{array}{c} z_{a,1}\, \overline{z}_{c,1} + \cdots + z_{a,n}\, \overline{z}_{n,c} \\ +u_{a,1}\, \overline{v}_{c,1} + \cdots + u_{a,m}\, \overline{v}_{c,m} \\ +v_{a,1}\, \overline{u}_{c,1} + \cdots + v_{a,m}\, \overline{u}_{c,m} \end{array} \right\}.$$

- **Question :** *What are the infinitesimal CR symmetries ?*

$1 \leq i \leq e \quad 1 \leq j \leq n$

$$L_{z_{ij}z_{ij}} = \sum_{e'=1}^{e} \sum_{n' \neq j} \left(2 z_{e'j} z_{in'}\right)\partial_{z_{e'n'}} + \sum_{1 \leq e' < i} \left(2 z_{e'j} z_{ij} - w_{e'i} - w_{ie'}\right)\partial_{z_{e'j}} + \left(2 z_{ij}^2 - w_{ii}\right)\partial_{z_{ij}} + \sum_{i < e' \leq e} \left(2 z_{e'j} z_{ij} - w_{e'i} + w_{ie'}\right)\partial_{z_{e'j}}$$

$$+ \sum_{e'=1}^{e} \sum_{m'=1}^{m} \left(2 z_{e'j} u_{im'}\right)\partial_{u_{e'm'}} + \sum_{e'} \sum_{m'} \left(2 z_{e'j} u_{im'}\right)\partial_{v_{e'm'}}$$

$$+ \sum_{1 \leq n < s < i} \left(z_{nj}\left[-w_{si} + w_{is}\right] + z_{sj}\left[w_{ni} - w_{in}\right]\right)\partial_{w_{ns}} + \sum_{1 \leq n < i} \left(z_{nj}\left[w_{ii}\right] + z_{ij}\left[w_{ni} - w_{in}\right]\right)\partial_{w_{ni}} + \sum_{1 \leq n \leq i} \left(z_{nj}\left[w_{is} + w_{si}\right] + z_{sj}\left[w_{ni} - w_{in}\right]\right)\partial_{w_{ns}}$$

$$+ \sum_{i < n \leq e} \left(z_{ij}\left[w_{is} + w_{si}\right] + z_{sj}\left[-w_{ii}\right]\right)\partial_{w_{is}} + \sum_{i < n < s \leq e} \left(z_{nj}\left[w_{is} + w_{si}\right] + z_{sj}\left[-w_{in} - w_{in}\right]\right)\partial_{w_{ns}}$$

$$+ \sum_{1 \leq n < i} \left(z_{nj}\left[-2w_{ni} + 2w_{in}\right]\right)\partial_{w_{nn}} + \left(z_{ij}\left[2w_{ij}\right]\right)\partial_{w_{ii}} + \sum_{i < n \leq e} \left(z_{nj}\left[2w_{in} + 2w_{ni}\right]\right)\partial_{w_{nn}}$$

$$+ \sum_{1 \leq s < n < i} \left(z_{nj}\left[-w_{si} + w_{is}\right] + z_{sj}\left[-w_{ni} + w_{in}\right]\right)\partial_{w_{ns}} + \sum_{1 \leq s < i} \left(z_{ij}\left[-w_{si} + w_{is}\right] + z_{sj}\left[w_{ii}\right]\right)\partial_{w_{is}} + \sum_{i < n \leq e} \left(z_{nj}\left[-w_{si} + w_{is}\right] + z_{sj}\left[w_{in} + w_{ni}\right]\right)\partial_{w_{ns}}$$

$$+ \sum_{i < n \leq e} \left(z_{nj}\left[w_{ii}\right] + z_{ij}\left[w_{in} + w_{ni}\right]\right)\partial_{w_{ni}} + \sum_{i < s < n \leq e} \left(z_{nj}\left[w_{is} + w_{si}\right] + z_{sj}\left[w_{in} + w_{ni}\right]\right)\partial_{w_{ns}}$$

19 SUMS

$\partial w_{n<s}$

∂w_{nn}

$\partial w_{n>s}$

∂w_{ni}

$1 \leq i \leq e \quad 1 \leq j \leq n$

$$I L_{z_{ij}z_{ij}} = \sum_{e'} \sum_{n' \neq j} \left(2 I z_{e'j} z_{in'}\right)\partial_{z_{e'n'}} + \sum_{1 \leq e' < i} \left(2 I z_{e'j} z_{ij} + I w_{e'i} + I w_{ie'}\right)\partial_{z_{e'j}} + \left(2 I z_{ij}^2 + I w_{ij}\right)\partial_{z_{ij}} + \sum_{i < e' \leq e} \left(2 I z_{e'j} z_{ij} + I w_{e'i} - I w_{ie'}\right)\partial_{z_{e'j}}$$

$$+ \sum_{e'=1}^{e} \sum_{m'=1}^{m} \left(I 2 z_{e'j} u_{im'}\right)\partial_{u_{e'm'}} + \sum_{e'} \sum_{m'} \left(I 2 z_{e'j} u_{im'}\right)\partial_{v_{e'm'}}$$

$$+ I\left(\bigcirc\right)$$

19 SUMS

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$1\le i<j\le \ell$

$$\underline{L_{w_{ij}w_{ij}}} := \sum_{1\le \ell'<i}\sum_{n'=1}^{n}\left(z_{in'}[w_{j\ell'}+w_{\ell' j}]+z_{jn'}[-w_{i\ell'}-w_{\ell' i}]\right)\partial z_{\ell' n'}$$

$$+\sum_{n'=1}^{n}\left(z_{in'}[w_{ij}+w_{ji}]+z_{jn'}[-w_{ii}]\right)\partial z_{in'}$$

$$+\sum_{i<\ell'<j}\sum_{n'=1}^{n}\left(z_{in'}[w_{j\ell'}+w_{\ell' j}]+z_{jn'}[w_{i\ell'}-w_{\ell' i}]\right)\partial z_{\ell' n'}$$

$$+\sum_{n'=1}^{n}\left(z_{in'}[w_{jj}]+z_{jn'}[w_{ij}-w_{ji}]\right)\partial z_{jn'}$$

$$+\sum_{j<\ell'\le \ell}\sum_{n'=1}^{n}\left(z_{in'}[-w_{j\ell'}+w_{\ell' j}]+z_{jn'}[w_{i\ell'}-w_{\ell' i}]\right)\partial z_{\ell' n'}$$

$$+\sum_{1\le \ell'<i}\sum_{m'=1}^{m}\left(u_{im'}[w_{j\ell'}+w_{\ell' j}]+u_{jm'}[-w_{i\ell'}-w_{\ell' i}]\right)\partial u_{\ell' m'}$$

$$+\sum_{m'=1}^{m}\left(u_{im'}[w_{ij}+w_{ji}]+u_{jm'}[-w_{ii}]\right)\partial u_{im'}$$

$$+\sum_{i<\ell'<j}\sum_{m'=1}^{m}\left(u_{im'}[w_{j\ell'}+w_{\ell' j}]+u_{jm'}[w_{i\ell'}-w_{\ell' i}]\right)\partial u_{\ell' m'}$$

$$+\sum_{m'=1}^{m}\left(u_{im'}[w_{jj}]+u_{jm'}[w_{ij}-w_{ji}]\right)\partial u_{jm'}$$

$$+\sum_{j<\ell'\le \ell}\sum_{m'=1}^{m}\left(u_{im'}[-w_{j\ell'}+w_{\ell' j}]+u_{jm'}[w_{i\ell'}-w_{\ell' i}]\right)\partial u_{\ell' m'}$$

$$+\sum_{1\le e'<i}\sum_{m'=1}^{m}\left(v_{im'}\left[w_{je'}+w_{e'j}\right]+v_{jm'}\left[-w_{ie'}-w_{e'i}\right]\right)\partial v_{e'm'}$$

$$+\sum_{m'=1}^{m}\left(v_{im'}\left[w_{ij}+w_{ji}\right]+v_{jm'}\left[-w_{ii}\right]\right)\partial v_{im'}$$

$$+\sum_{i<e'<j}\sum_{m'=1}^{m}\left(v_{im'}\left[w_{je'}+w_{e'j}\right]+v_{jm'}\left[w_{ie'}-w_{e'i}\right]\right)\partial v_{e'm'}$$

$$+\sum_{m'=1}^{m}\left(v_{im'}\left[w_{jj}\right]+v_{jm'}\left[w_{ij}-w_{ji}\right]\right)\partial v_{jm'}$$

$$+\sum_{j<e'\le e}\sum_{m'=1}^{m}\left(v_{im'}\left[-w_{je'}+w_{e'j}\right]+v_{jm'}\left[w_{ie'}-w_{e'i}\right]\right)\partial v_{e'm'}$$

↓ Better order: 1, 6, 7, 8, 2, 17, 19, 12, 14, 3, 18, 9, 15, 16, 4, 13, 10, 11, 5

1 $$+\sum_{1\le n<i}\left(-2w_{ni}w_{jn}+2w_{nj}w_{in}\right)\partial w_{nn}$$

2 $$+\sum_{1\le n<i}\left(w_{ni}w_{ij}-w_{in}w_{ji}+w_{ii}w_{jn}\right)\partial w_{ni}$$

3 $$+\sum_{1\le n<i}\left(w_{nj}w_{ij}-w_{in}w_{jj}+w_{ji}w_{jn}\right)\partial w_{nj}$$

4 $$+\sum_{1\le s<i}\left(-w_{si}w_{ji}+w_{sj}w_{ii}+w_{is}w_{ij}\right)\partial w_{is}$$

5 $$+\left(2w_{ii}w_{ij}\right)\partial w_{ii}$$

6 $$+\sum_{i<s<j}\left(-w_{ii}w_{js}+w_{is}w_{ij}+w_{si}w_{ji}\right)\partial w_{is}$$

7 $$+\left(-w_{ii}w_{jj}+w_{ij}^{2}+w_{ji}^{2}\right)\partial w_{ij}$$

8
$$+ \sum_{j<s\leq e} \left(-w_{ii}w_{sj} + w_{ij}w_{is} + w_{ji}w_{si}\right) \partial w_{is}$$

9
$$+ \sum_{i<n<j} \left(w_{ii}w_{nj} + w_{ji}w_{in} + w_{ij}w_{ni}\right) \partial w_{ni}$$

10
$$+ \sum_{i<n<j} \left(2w_{in}w_{jn} + 2w_{ni}w_{nj}\right) \partial w_{nn}$$

11
$$+ \sum_{i<n<j} \left(w_{ij}w_{nj} - w_{ni}w_{jj} + w_{ji}w_{jn}\right) \partial w_{nj}$$

12
$$+ \sum_{1\leq s<i} \left(-w_{si}w_{jj} + w_{sj}w_{ji} + w_{ij}w_{js}\right) \partial w_{js}$$

13
$$+ \left(2w_{ij}w_{ji}\right) \partial w_{ji}$$

14
$$+ \sum_{i<s<j} \left(w_{is}w_{jj} + w_{ij}w_{js} + w_{sj}w_{ji}\right) \partial w_{js}$$

15
$$+ \sum_{j<n\leq e} \left(2w_{in}w_{nj} - 2w_{jn}w_{ni}\right) \partial w_{nn}$$

16
$$+ \sum_{j<s\leq e} \left(w_{ij}w_{js} - w_{ji}w_{sj} + w_{jj}w_{si}\right) \partial w_{js}$$

SWAP

17
$$+ \sum_{j<n\leq e} \left(-w_{ii}w_{jn} + w_{ij}w_{ni} + w_{in}w_{ji}\right) \partial w_{ni}$$

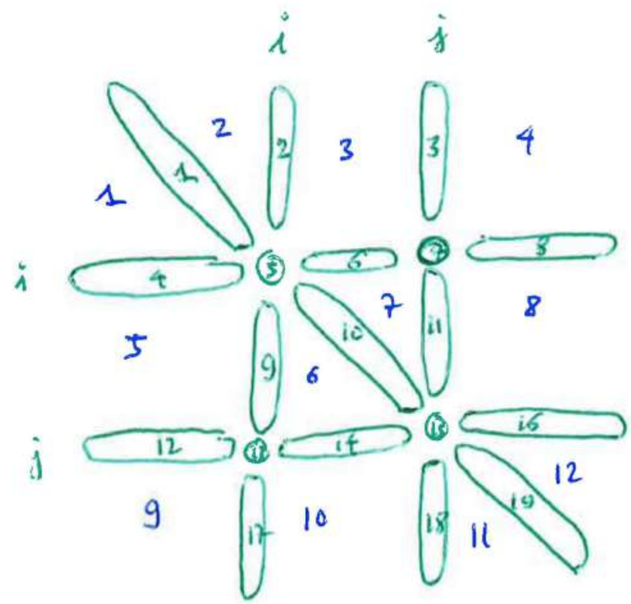
18
$$+ \sum_{j<n\leq e} \left(w_{ij}w_{nj} + w_{in}w_{jj} - w_{ji}w_{jn}\right) \partial w_{nj}$$

19
$$+ \left(2w_{ij}w_{ji}\right) \partial w_{jj}$$

← NOT FINISHED : 12 SUMS REMAIN

BETTER ORDER: 29, 20, 21, 22, 28, 30, 31, 23, 27, 26, 25, 24

1=1
$$+ \sum_{1\leq s<n<i} \left(-w_{si}w_{jn} + w_{sj}w_{in} - w_{ni}w_{js} + w_{nj}w_{is}\right) \partial w_{ns}$$



-106-

$\mathfrak{su}(p,q)$ CR Models : Expected dimensions

From Zhaohu's construction :

$$\begin{aligned}
\dim \mathfrak{g}_{-2} &= \ell^2 = \operatorname{codim} M, \\
\dim \mathfrak{g}_{-1} &= 2\,(n+m+m)\,\ell = \operatorname{CRdim} M, \\
\dim \mathfrak{g}_{0} &= \dim \mathfrak{su}(p,q) - 2\operatorname{codim} M - 2\operatorname{CRdim} M, \\
\dim \mathfrak{g}_{1} &= \dim \mathfrak{g}_{-1} \\
\dim \mathfrak{g}_{2} &= \dim \mathfrak{g}_{-2},
\end{aligned}$$

hence :

$$\begin{aligned}
\dim \mathfrak{g}_{0} &= \dim \mathfrak{su}(p,q) - 2\,codimR - 2\,CRdim \\
&= (p+q)^2 - 1 - 2\,\ell^2 - 4\,(n+m+m)\,\ell \\
&= (n+m+m)^2 + 2\,\ell^2 - 1.
\end{aligned}$$

1. $\mathfrak{su}(p,q)$ CR Models : Generators of $\mathfrak{g}_{-2}$ and of $\mathfrak{g}_{-1}$

Simply :

$$\mathfrak{g}_{-2} = \operatorname{Span}\Big\{\partial_{w_{\underset{<}{ac}}},\ \sqrt{-1}\,\partial_{w_{\underset{\geqslant}{ac}}}\Big\}$$

Next, $\mathfrak{g}_{-1}$ is generated by :

$$\begin{aligned}
L_{z_{ij}} := \partial_{z_{ij}} &+ \sum_{1\leqslant r<i} z_{rj}\,\partial_{w_{ri}} + \sum_{1\leqslant s<i} z_{sj}\,\partial_{w_{is}} + 2\,z_{ij}\,\partial_{w_{ii}} \\
&+ \sum_{i<r\leqslant \ell} z_{rj}\,\partial_{w_{ri}} - \sum_{i<s\leqslant \ell} z_{sj}\,\partial_{w_{is}} \qquad (1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant n),
\end{aligned}$$

$$IL_{z_{ij}} = \sqrt{-1}\bigg(\partial_{z_{ij}} - \sum_{1\leqslant r<i} z_{rj}\,\partial_{w_{ri}} - \sum_{1\leqslant s<i} z_{sj}\,\partial_{w_{is}} - 2\,z_{ij}\,\partial_{w_{ii}}$$

$$- \sum_{i<r\leqslant\ell} z_{rj}\,\partial_{w_{ri}} + \sum_{i<s\leqslant\ell} z_{sj}\,\partial_{w_{is}}\bigg) \qquad (1\leqslant i\leqslant\ell,\ 1\leqslant j\leqslant n),$$

$$L_{u_{ik}} = \partial_{u_{ik}} + \sum_{1\leqslant r<i} v_{rk}\,\partial_{w_{ri}} + \sum_{1\leqslant s<i} v_{sk}\,\partial_{w_{is}} + 2\,v_{ik}\,\partial_{w_{ii}}$$

$$+ \sum_{i<r\leqslant\ell} v_{rk}\,\partial_{w_{ri}} - \sum_{i<s\leqslant\ell} v_{sk}\,\partial_{w_{is}} \qquad (1\leqslant i\leqslant\ell,\ 1\leqslant k\leqslant m),$$

$$IL_{u_{ik}} = \sqrt{-1}\bigg(\partial_{u_{ik}} - \sum_{1\leqslant r<i} v_{rk}\,\partial_{w_{ri}} - \sum_{1\leqslant s<i} v_{sk}\,\partial_{w_{is}} - 2\,v_{ik}\,\partial_{w_{ii}}$$

$$- \sum_{i<r\leqslant\ell} v_{rk}\,\partial_{w_{ri}} + \sum_{i<s\leqslant\ell} v_{sk}\,\partial_{w_{is}}\bigg) \qquad (1\leqslant i\leqslant\ell,\ 1\leqslant k\leqslant m),$$

$$L_{v_{ik}} = \partial_{v_{ik}} + \sum_{1\leqslant r<i} u_{rk}\,\partial_{w_{ri}} + \sum_{1\leqslant s<i} u_{sk}\,\partial_{w_{is}} + 2\,u_{ik}\,\partial_{w_{ii}}$$

$$+ \sum_{i<r\leqslant\ell} u_{rk}\,\partial_{w_{ri}} - \sum_{i<s\leqslant\ell} u_{sk}\,\partial_{w_{is}} \qquad (1\leqslant i\leqslant\ell,\ 1\leqslant k\leqslant m),$$

$$IL_{v_{ik}} = \sqrt{-1}\bigg(\partial_{v_{ik}} - \sum_{1\leqslant r<i} u_{rk}\,\partial_{w_{ri}} - \sum_{1\leqslant s<i} u_{sk}\,\partial_{w_{is}} - 2\,u_{ik}\,\partial_{w_{ii}}$$

$$- \sum_{i<r\leqslant\ell} u_{rk}\,\partial_{w_{ri}} + \sum_{i<s\leqslant\ell} u_{sk}\,\partial_{w_{is}}\bigg) \qquad (1\leqslant i\leqslant\ell,\ 1\leqslant k\leqslant m).$$

2. $\mathfrak{su}(p,q)$ CR Models : Generators of $\mathfrak{g}_0$

With the decomposition :

$$\begin{aligned}\big(n+m+m\big)^2 \;=\;& m^2+\tfrac{m(m-1)}{2}+\tfrac{m(m-1)}{2}+n\,m+n\,m+\tfrac{n(n-1)}{2}\\ &+m^2+\tfrac{m(m-1)}{2}+\tfrac{m(m-1)}{2}+n\,m+n\,m+\tfrac{n(n-1)}{2}\\ &+n+m+m,\end{aligned}$$

there are $6+6+3$ families of generators for $\mathfrak{g}_0$ that are independent of any $\partial_{w_{\ell'\ell''}}$.

Firstly :

$$L^1_{k_1k_2} := \sum_{\ell'=1}^{\ell} u_{\ell'k_1}\,\partial_{u_{\ell'k_2}} - \sum_{\ell'=1}^{\ell} v_{\ell'k_2}\,\partial_{v_{\ell'k_1}} \qquad (1\leqslant k_1, k_2\leqslant m),$$

$$L^2_{k_1k_2} := \sum_{\ell'=1}^{\ell} v_{\ell'k_1}\,\partial_{u_{\ell'k_2}} - \sum_{\ell'=1}^{\ell} v_{\ell'k_2}\,\partial_{u_{\ell'k_1}} \qquad (1\leqslant k_1 < k_2\leqslant m),$$

$$L^3_{k_1k_2} := \sum_{\ell'=1}^{\ell} u_{\ell'k_1}\,\partial_{v_{\ell'k_2}} - \sum_{\ell'=1}^{\ell} u_{\ell'k_2}\,\partial_{v_{\ell'k_1}} \qquad (1\leqslant k_1 < k_2\leqslant m),$$

$$L^4_{i,k} := \sum_{\ell'=1}^{\ell} u_{\ell'k}\,\partial_{z_{\ell'i}} - \sum_{\ell'=1}^{\ell} z_{\ell'i}\,\partial_{v_{\ell'k}} \qquad (1\leqslant i\leqslant n,\ 1\leqslant k\leqslant m),$$

$$L^5_{i,k} := \sum_{\ell'=1}^{\ell} v_{\ell'k}\,\partial_{z_{\ell'i}} - \sum_{\ell'=1}^{\ell} z_{\ell'i}\,\partial_{u_{\ell'k}} \qquad (1\leqslant i\leqslant n,\ 1\leqslant k\leqslant m),$$

$$L^6_{i_1,i_2} := \sum_{\ell'=1}^{\ell} z_{\ell'i_1}\,\partial_{z_{\ell'i_2}} - \sum_{\ell'=1}^{\ell} z_{\ell'i_2}\,\partial_{z_{\ell'i_1}} \qquad (1\leqslant i_1 < i_2\leqslant n),$$

Secondly :

$$
\begin{aligned}
IL^1_{k_1k_2} &:= \sqrt{-1}\sum_{\ell'=1}^{\ell} u_{\ell' k_1}\,\partial_{u_{\ell' k_2}} + \sqrt{-1}\sum_{\ell'=1}^{\ell} v_{\ell' k_2}\,\partial_{v_{\ell' k_1}} && (1\leqslant k_1, k_2\leqslant m),\\
IL^2_{k_1k_2} &:= \sqrt{-1}\sum_{\ell'=1}^{\ell} v_{\ell' k_1}\,\partial_{u_{\ell' k_2}} + \sqrt{-1}\sum_{\ell'=1}^{\ell} v_{\ell' k_2}\,\partial_{u_{\ell' k_1}} && (1\leqslant k_1 < k_2\leqslant m),\\
IL^3_{k_1k_2} &:= \sqrt{-1}\sum_{\ell'=1}^{\ell} u_{\ell' k_1}\,\partial_{v_{\ell' k_2}} + \sqrt{-1}\sum_{\ell'=1}^{\ell} u_{\ell' k_2}\,\partial_{v_{\ell' k_1}} && (1\leqslant k_1 < k_2\leqslant m),\\
IL^4_{i,k} &:= \sqrt{-1}\sum_{\ell'=1}^{\ell} u_{\ell' k}\,\partial_{z_{\ell' i}} + \sqrt{-1}\sum_{\ell'=1}^{\ell} z_{\ell' i}\,\partial_{v_{\ell' k}} && (1\leqslant i\leqslant n,\ 1\leqslant k\leqslant m),\\
IL^5_{i,k} &:= \sqrt{-1}\sum_{\ell'=1}^{\ell} v_{\ell' k}\,\partial_{z_{\ell' i}} + \sqrt{-1}\sum_{\ell'=1}^{\ell} z_{\ell' i}\,\partial_{u_{\ell' k}} && (1\leqslant i\leqslant n,\ 1\leqslant k\leqslant m),\\
IL^6_{i_1,i_2} &:= \sqrt{-1}\sum_{\ell'=1}^{\ell} z_{\ell' i_1}\,\partial_{z_{\ell' i_2}} + \sqrt{-1}\sum_{\ell'=1}^{\ell} z_{\ell' i_2}\,\partial_{z_{\ell' i_1}} && (1\leqslant i_1 < i_2\leqslant n),
\end{aligned}
$$

Thirdly :

$$IL_i^7 := \sqrt{-1}\sum_{\ell'=1}^{\ell} z_{\ell' i}\,\partial_{z_{\ell' i}} \qquad\qquad (1 \leqslant i \leqslant n),$$

$$IL_k^8 := \sqrt{-1}\sum_{\ell'=1}^{\ell} v_{\ell' k}\,\partial_{u_{\ell' k}} \qquad\qquad (1 \leqslant k \leqslant m),$$

$$IL_k^9 := \sqrt{-1}\sum_{\ell'=1}^{\ell} u_{\ell' k}\,\partial_{v_{\ell' k}} \qquad\qquad (1 \leqslant k \leqslant m).$$

In addition, there are 5 families of generators which do depend on some $\partial_{w_{\ell'\ell'}}$:

$$\begin{aligned}
&L_{i,j} && (1 \leqslant i < j \leqslant \ell),\\
&L_{i,j} && (1 \leqslant j \leqslant i \leqslant \ell),\\
&IL_{i,j} && (1 \leqslant i < j \leqslant \ell),\\
&IL_{i,i} && (1 \leqslant i \leqslant \ell),\\
&IL_{i,j} && (1 \leqslant j < i \leqslant \ell).
\end{aligned}$$

For $1 \leqslant i < j \leqslant \ell$:

$$\begin{aligned}
L_{i,j} := &\sum_{n'=1}^{n} z_{in'}\,\partial_{z_{jn'}} + \sum_{m'=1}^{m} u_{im'}\,\partial_{u_{jm'}} + \sum_{m'=1}^{m} v_{im'}\,\partial_{v_{jm'}} \\
&+ \sum_{1 \leqslant s < i} w_{is}\,\partial_{w_{js}} + \sum_{i \leqslant s \leqslant j} w_{si}\,\partial_{w_{js}} + \sum_{j < s \leqslant \ell} w_{is}\,\partial_{w_{js}} \\
&+ \sum_{1 \leqslant r < i} w_{ri}\,\partial_{w_{rj}} - \sum_{i < r < j} w_{ir}\,\partial_{w_{rj}} + \sum_{j \leqslant r \leqslant \ell} w_{ri}\,\partial_{w_{rj}}.
\end{aligned}$$

For $1 \leqslant j \leqslant i \leqslant \ell$:

$$
\begin{aligned}
L_{i,j} \;:=\; & \sum_{n'=1}^{n} z_{in'}\,\partial_{z_{jn'}} + \sum_{m'=1}^{m} u_{im'}\,\partial_{u_{jm'}} + \sum_{m'=1}^{m} v_{im'}\,\partial_{v_{jm'}} \\
& + \sum_{1\leqslant s\leqslant j} w_{is}\,\partial_{w_{js}} - \sum_{j<s<i} w_{si}\,\partial_{w_{js}} + \sum_{i<s\leqslant \ell} w_{is}\,\partial_{w_{js}} \\
& + \sum_{1\leqslant r<j} w_{ri}\,\partial_{w_{rj}} + \sum_{j\leqslant r\leqslant i} w_{ir}\,\partial_{w_{rj}} + \sum_{i<r\leqslant \ell} w_{ri}\,\partial_{w_{rj}}.
\end{aligned}
$$

For $1 \leqslant i < j \leqslant \ell$:

$$
\begin{aligned}
IL_{i,j} \;:=\; & \sqrt{-1}\sum_{n'=1}^{n} z_{in'}\,\partial_{z_{jn'}} + \sqrt{-1}\sum_{m'=1}^{m} u_{im'}\,\partial_{u_{jm'}} + \sqrt{-1}\sum_{m'=1}^{m} v_{im'}\,\partial_{v_{jm'}} \\
& - \sqrt{-1}\sum_{1\leqslant s<i} w_{si}\,\partial_{w_{js}} + \sqrt{-1}\sum_{i<s\leqslant j} w_{is}\,\partial_{w_{js}} + \sqrt{-1}\sum_{j<s\leqslant \ell} w_{si}\,\partial_{w_{js}} \\
& - \sqrt{-1}\sum_{1\leqslant r\leqslant i} w_{ir}\,\partial_{w_{rj}} - \sqrt{-1}\sum_{i<r<j} w_{ri}\,\partial_{w_{rj}} + \sqrt{-1}\sum_{j\leqslant r\leqslant \ell} w_{ir}\,\partial_{w_{rj}}.
\end{aligned}
$$

For $1 \leqslant i \leqslant \ell$:

$$
\begin{aligned}
IL_{i,i} \;:=\; & \sqrt{-1}\sum_{n'=1}^{n} z_{in'}\,\partial_{z_{in'}} + \sqrt{-1}\sum_{m'=1}^{m} u_{im'}\,\partial_{u_{im'}} + \sqrt{-1}\sum_{m'=1}^{m} v_{im'}\,\partial_{v_{im'}} \\
& - \sqrt{-1}\sum_{1\leqslant s<i} w_{si}\,\partial_{w_{is}} + \sqrt{-1}\sum_{i<s\leqslant \ell} w_{si}\,\partial_{w_{is}} \\
& - \sqrt{-1}\sum_{1\leqslant s<i} w_{is}\,\partial_{w_{si}} + \sqrt{-1}\sum_{i<s\leqslant \ell} w_{si}\,\partial_{w_{is}}.
\end{aligned}
$$

For $1 \leqslant j < i \leqslant \ell$:

$$
\begin{aligned}
IL_{i,j} := \sqrt{-1} \sum_{n'=1}^{n} z_{in'}\, \partial_{z_{jn'}} &+ \sqrt{-1} \sum_{m'=1}^{m} u_{im'}\, \partial_{u_{jm'}} + \sqrt{-1} \sum_{m'=1}^{m} v_{im'}\, \partial_{v_{jm'}} \\
&- \sqrt{-1} \sum_{1 \leqslant s \leqslant j} w_{si}\, \partial_{w_{js}} + \sqrt{-1} \sum_{j < s \leqslant i} w_{is}\, \partial_{w_{js}} + \sqrt{-1} \sum_{i < s \leqslant \ell} w_{si}\, \partial_{w_{js}} \\
&- \sqrt{-1} \sum_{1 \leqslant r < j} w_{ir}\, \partial_{w_{rj}} - \sqrt{-1} \sum_{j \leqslant r < i} w_{ri}\, \partial_{w_{rj}} + \sqrt{-1} \sum_{i < r \leqslant \ell} w_{ir}\, \partial_{w_{rj}}.
\end{aligned}
$$

3. $\mathfrak{su}(p,q)$ CR Models : Generators of $\mathfrak{g}_1$

There are 6 families of generators for $\mathfrak{g}_1$:

$$
\begin{aligned}
&L_{z_{ij}z_{ij}} && (1 \leqslant i \leqslant \ell,\, 1 \leqslant j \leqslant n), \\
&IL_{z_{ij}z_{ij}} && (1 \leqslant i \leqslant \ell,\, 1 \leqslant j \leqslant n), \\
&L_{u_{ik}u_{ik}} && (1 \leqslant i \leqslant \ell,\, 1 \leqslant k \leqslant m), \\
&IL_{u_{ik}u_{ik}} && (1 \leqslant i \leqslant \ell,\, 1 \leqslant k \leqslant m), \\
&L_{v_{ik}v_{ik}} && (1 \leqslant i \leqslant \ell,\, 1 \leqslant k \leqslant m), \\
&IL_{v_{ik}f_{ik}} && (1 \leqslant i \leqslant \ell,\, 1 \leqslant k \leqslant m).
\end{aligned}
$$

For $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant n$:

$$
\begin{aligned}
L_{z_{ij}z_{ij}} := & \sum_{\ell'=1}^{\ell} \sum_{\substack{n'=1 \\ n'\neq j}}^{n} \big(2\, z_{\ell' j}\, z_{in'}\big)\, \partial_{z_{\ell' n'}} \\
& + \sum_{1\leqslant \ell' < i} \big(2\, z_{\ell' j}\, z_{ij} - w_{\ell' i} - w_{i\ell'}\big)\, \partial_{z_{\ell' j}} + \big(2\, z_{ij}^2 - w_{ii}\big)\, \partial_{z_{ij}} + \sum_{i<\ell'\leqslant \ell} \big(2\, z_{\ell' j}\, z_{ij} - w_{\ell' i} + w_{i\ell'}\big)\, \partial_{z_{\ell' j}} \\
& + \sum_{\ell'=1}^{\ell} \sum_{m'=1}^{m} \big(2\, z_{\ell' j}\, u_{im'}\big)\, \partial_{u_{\ell' m'}} + \sum_{\ell'=1}^{\ell} \sum_{m'=1}^{m} \big(2\, z_{\ell' j}\, u_{im'}\big)\, \partial_{v_{\ell' m'}} \\
& + \sum_{1\leqslant r<s<i} \Big(z_{rj}\, \big[-w_{si} + w_{is}\big] + z_{sj}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sum_{1\leqslant r<i} \Big(z_{rj}\, \big[w_{ii}\big] + z_{ij}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{ri}} \\
& + \sum_{1\leqslant r\leqslant i} \Big(z_{rj}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sum_{i<r\leqslant \ell} \Big(z_{ij}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[-w_{ii}\big]\Big)\, \partial_{w_{is}} \\
& + \sum_{i<r<s\leqslant \ell} \Big(z_{rj}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[-w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} \\
& + \sum_{1\leqslant r<i} \Big(z_{rj}\, \big[-2\, w_{ri} + 2\, w_{ir}\big]\Big)\, \partial_{w_{rr}} + \Big(z_{ij}\, \big[2\, w_{ii}\big]\Big)\, \partial_{w_{ii}} + \sum_{i<r\leqslant \ell} \Big(z_{rj}\, \big[2\, w_{ir} + 2\, w_{ri}\big]\Big)\, \partial_{w_{rr}} \\
& + \sum_{1\leqslant s<r<i} \Big(z_{rj}\, \big[-w_{si} + w_{is}\big] + z_{sj}\, \big[-w_{ri} + w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sum_{1\leqslant s<i} \Big(z_{ij}\, \big[-w_{si} + w_{is}\big] + w_{sj}\, \big[w_{ii}\big]\Big)\, \partial_{w_{is}} \\
& + \sum_{i<r\leqslant \ell} \Big(z_{rj}\, \big[-w_{si} + w_{is}\big] + z_{sj}\, \big[w_{ir} + w_{ri}\big]\Big)\, \partial_{w_{rs}} + \sum_{i<r\leqslant \ell} \Big(z_{rj}\, \big[w_{ii}\big] + z_{ij}\, \big[w_{ir} + w_{ri}\big]\Big)\, \partial_{w_{ri}} \\
& + \sum_{i<s<r\leqslant \ell} \Big(z_{rj}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[w_{ir} + w_{ri}\big]\Big)\, \partial_{w_{rs}}.
\end{aligned}
$$

For $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant n$:

$$
\begin{aligned}
IL_{z_{ij}z_{ij}} \;:=\; & \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{\substack{n'=1 \\ n'\neq j}}^{n} \big(2\, z_{\ell' j}\, z_{in'}\big)\, \partial_{z_{\ell' n'}} \\
& + \sqrt{-1} \sum_{1\leqslant \ell' < i} \big(2\, z_{\ell' j}\, z_{ij} + w_{\ell' i} + w_{i\ell'}\big)\, \partial_{z_{\ell' j}} + \sqrt{-1}\, \big(2\, z_{ij}^2 + w_{ii}\big)\, \partial_{z_{ij}} \\
& + \sqrt{-1} \sum_{i < \ell' \leqslant \ell} \big(2\, z_{\ell' j}\, z_{ij} + w_{\ell' i} - w_{i\ell'}\big)\, \partial_{z_{\ell' j}} \\
& + \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{m'=1}^{m} \big(2\, z_{\ell' j}\, u_{im'}\big)\, \partial_{u_{\ell' m'}} + \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{m'=1}^{m} \big(2\, z_{\ell' j}\, u_{im'}\big)\, \partial_{v_{\ell' m'}}
\end{aligned}
$$

$$
\begin{aligned}
& + \sqrt{-1} \sum_{1\leqslant r < s < i} \Big(z_{rj}\, \big[-w_{si} + w_{is}\big] + z_{sj}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sqrt{-1} \sum_{1\leqslant r < i} \Big(z_{rj}\, \big[w_{ii}\big] + z_{ij}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{ri}} \\
& + \sqrt{-1} \sum_{1\leqslant r \leqslant i} \Big(z_{rj}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sqrt{-1} \sum_{i < r \leqslant \ell} \Big(z_{ij}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[-w_{ii}\big]\Big)\, \partial_{w_{is}} \\
& + \sqrt{-1} \sum_{i < r < s \leqslant \ell} \Big(z_{rj}\, \big[w_{is} + w_{si}\big] + z_{sj}\, \big[-w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}}
\end{aligned}
$$

$$
+ \sqrt{-1} \sum_{1\leqslant r < i} \Big(z_{rj}\, \big[-2\, w_{ri} + 2\, w_{ir}\big]\Big)\, \partial_{w_{rr}} + \Big(z_{ij}\, \big[2\, w_{ii}\big]\Big)\, \partial_{w_{ii}} + \sqrt{-1} \sum_{i < r \leqslant \ell} \Big(z_{rj}\, \big[2\, w_{ir} + 2\, w_{ri}\big]\Big)\, \partial_{w_{rr}}
$$

$$+\sqrt{-1}\sum_{1\leqslant s<r<i}\Big(z_{rj}\,\big[-w_{si}+w_{is}\big]+z_{sj}\,\big[-w_{ri}+w_{ir}\big]\Big)\,\partial_{w_{rs}}$$
$$+\sqrt{-1}\sum_{1\leqslant s<i}\Big(z_{ij}\,\big[-w_{si}+w_{is}\big]+w_{sj}\,\big[w_{ii}\big]\Big)\,\partial_{w_{is}}$$
$$+\sqrt{-1}\sum_{i<r\leqslant\ell}\Big(z_{rj}\,\big[-w_{si}+w_{is}\big]+z_{sj}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}+\sqrt{-1}\sum_{i<r\leqslant\ell}\Big(z_{rj}\,\big[w_{ii}\big]+z_{ij}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{ri}}$$
$$+\sqrt{-1}\sum_{i<s<r\leqslant\ell}\Big(z_{rj}\,\big[w_{is}+w_{si}\big]+z_{sj}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}.$$

For $1\leqslant i\leqslant\ell$ and $1\leqslant k\leqslant m$:

$$\begin{aligned}L_{u_{ik}u_{ik}}:=&\sum_{\ell'=1}^{\ell}\sum_{n'=1}^{n}\big(2\,z_{in'}\,u_{\ell'k}\big)\,\partial_{z_{\ell'n'}}\\&+\sum_{\ell'=1}^{\ell}\sum_{m'=1}^{m}\big(2\,u_{im'}\,u_{\ell'm'}\big)\,\partial_{u_{\ell'm'}}\\&+\sum_{\ell'=1}^{\ell}\sum_{\substack{m'=1\\m'\neq k}}^{m}\big(2\,u_{\ell'k}\,v_{im'}\big)\,\partial_{v_{\ell'm'}}\end{aligned}$$
$$+\sum_{1\leqslant\ell'<i}\big(2\,u_{\ell'k}\,v_{ik}-w_{\ell'i}-w_{i\ell'}\big)\,\partial_{v_{\ell'k}}+\big(2\,u_{ik}\,v_{ik}-w_{ii}\big)\,\partial_{v_{ik}}+\sum_{i<\ell'\leqslant\ell}\big(2\,u_{\ell'k}\,v_{ik}-w_{\ell'i}+w_{i\ell'}\big)\,\partial_{v_{\ell'k}}$$

$$
\begin{aligned}
&+ \sum_{1\leqslant r<s<i} \Big(u_{rk}\big[-w_{si}+w_{is}\big]+u_{sk}\big[w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{rs}} + \sum_{1\leqslant r<i}\Big(u_{rk}\big[w_{ii}\big]+u_{ik}\big[w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{ri}}\\
&+ \sum_{1\leqslant r\leqslant i}\Big(u_{rk}\big[w_{is}+w_{si}\big]+u_{sk}\big[w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{rs}} + \sum_{i<r\leqslant \ell}\Big(u_{ik}\big[w_{is}+w_{si}\big]+u_{sk}\big[-w_{ii}\big]\Big)\,\partial_{w_{is}}\\
&+ \sum_{i<r<s\leqslant\ell}\Big(u_{rk}\big[w_{is}+w_{si}\big]+u_{sk}\big[-w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{rs}}
\end{aligned}
$$

$$
+ \sum_{1\leqslant r<i}\Big(u_{rk}\big[-2\,w_{ri}+2\,w_{ir}\big]\Big)\,\partial_{w_{rr}} + \Big(u_{ik}\big[2\,w_{ii}\big]\Big)\,\partial_{w_{ii}} + \sum_{i<r\leqslant\ell}\Big(u_{rk}\big[2\,w_{ir}+2\,w_{ri}\big]\Big)\,\partial_{w_{rr}}
$$

$$
\begin{aligned}
&+ \sum_{1\leqslant s<r<i}\Big(u_{rk}\big[-w_{si}+w_{is}\big]+u_{sk}\big[-w_{ri}+w_{ir}\big]\Big)\,\partial_{w_{rs}} + \sum_{1\leqslant s<i}\Big(u_{ik}\big[-w_{si}+w_{is}\big]+w_{sk}\big[w_{ii}\big]\Big)\,\partial_{w_{is}}\\
&+ \sum_{i<r\leqslant\ell}\Big(u_{rk}\big[-w_{si}+w_{is}\big]+u_{sk}\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}} + \sum_{i<r\leqslant\ell}\Big(u_{rk}\big[w_{ii}\big]+u_{ik}\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{ri}}\\
&+ \sum_{i<s<r\leqslant\ell}\Big(u_{rk}\big[w_{is}+w_{si}\big]+u_{sk}\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}.
\end{aligned}
$$

For $1 \leqslant i \leqslant \ell$ and $1 \leqslant k \leqslant m$:

$$IL_{u_{ik}u_{ik}} := \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{n'=1}^{n} \big(2\, z_{in'}\, u_{\ell' k}\big)\, \partial_{z_{\ell' n'}}$$

$$+ \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{m'=1}^{m} \big(2\, u_{im'}\, u_{\ell' m'}\big)\, \partial_{u_{\ell' m'}}$$

$$+ \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{\substack{m'=1 \\ m' \neq k}}^{m} \big(2\, u_{\ell' k}\, v_{im'}\big)\, \partial_{v_{\ell' m'}}$$

$$\overline{1} \sum_{1 \leqslant \ell' < i} \big(2\, u_{\ell' k}\, v_{ik} + w_{\ell' i} + w_{i \ell'}\big)\, \partial_{v_{\ell' k}} + \sqrt{-1}\, \big(2\, u_{ik}\, v_{ik} + w_{ii}\big)\, \partial_{v_{ik}} + \sqrt{-1} \sum_{i < \ell' \leqslant \ell} \big(2\, u_{\ell' k}\, v_{ik} + w_{\ell' i} - w_{i \ell'}\big)\, \partial_{v_{\ell' k}}$$

$$+ \sqrt{-1} \sum_{1 \leqslant r < s < i} \Big(u_{rk}\, \big[-w_{si} + w_{is}\big] + u_{sk}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sqrt{-1} \sum_{1 \leqslant r < i} \Big(u_{rk}\, \big[w_{ii}\big] + u_{ik}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{ri}}$$

$$+ \sqrt{-1} \sum_{1 \leqslant r \leqslant i} \Big(u_{rk}\, \big[w_{is} + w_{si}\big] + u_{sk}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sqrt{-1} \sum_{i < r \leqslant \ell} \Big(u_{ik}\, \big[w_{is} + w_{si}\big] + u_{sk}\, \big[-w_{ii}\big]\Big)\, \partial_{w_{is}}$$

$$+ \sqrt{-1} \sum_{i < r < s \leqslant \ell} \Big(u_{rk}\, \big[w_{is} + w_{si}\big] + u_{sk}\, \big[-w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}}$$

$$+ \sqrt{-1} \sum_{1 \leqslant r < i} \Big(u_{rk}\, \big[-2\, w_{ri} + 2\, w_{ir}\big]\Big)\, \partial_{w_{rr}} + \Big(u_{ik}\, \big[2\, w_{ii}\big]\Big)\, \partial_{w_{ii}} + \sqrt{-1} \sum_{i < r \leqslant \ell} \Big(u_{rk}\, \big[2\, w_{ir} + 2\, w_{ri}\big]\Big)\, \partial_{w_{rr}}$$

$$+\sqrt{-1}\sum_{1\leqslant s<r<i}\Big(u_{rk}\left[-w_{si}+w_{is}\right]+u_{sk}\left[-w_{ri}+w_{ir}\right]\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{1\leqslant s<i}\Big(u_{ik}\left[-w_{si}+w_{is}\right]+w_{sk}\left[w_{ii}\right]\Big)\,\partial_{w_{is}}$$

$$+\sqrt{-1}\sum_{i<r\leqslant \ell}\Big(u_{rk}\left[-w_{si}+w_{is}\right]+u_{sk}\left[w_{ir}+w_{ri}\right]\Big)\,\partial_{w_{rs}}+\sqrt{-1}\sum_{i<r\leqslant \ell}\Big(u_{rk}\left[w_{ii}\right]+u_{ik}\left[w_{ir}+w_{ri}\right]\Big)\,\partial_{w_{ri}}$$

$$+\sqrt{-1}\sum_{i<s<r\leqslant \ell}\Big(u_{rk}\left[w_{is}+w_{si}\right]+u_{sk}\left[w_{ir}+w_{ri}\right]\Big)\,\partial_{w_{rs}}.$$

For $1\leqslant i\leqslant \ell$ and $1\leqslant k\leqslant m:$

$$L_{v_{ik}v_{ik}}:=\sum_{\ell'=1}^{\ell}\sum_{n'=1}^{n}\big(2\,z_{in'}\,v_{\ell' k}\big)\,\partial_{z_{\ell' n'}}$$

$$+\sum_{\ell'=1}^{\ell}\sum_{m'=1}^{m}\big(2\,v_{im'}\,v_{\ell' m'}\big)\,\partial_{v_{\ell' m'}}$$

$$+\sum_{\ell'=1}^{\ell}\sum_{\substack{m'=1\\ m'\neq k}}^{m}\big(2\,v_{\ell' k}\,v_{im'}\big)\,\partial_{v_{\ell' m'}}$$

$$+\sum_{1\leqslant \ell'<i}\big(2\,v_{\ell' k}\,v_{ik}-w_{\ell' i}-w_{i\ell'}\big)\,\partial_{v_{\ell' k}}+\big(2\,v_{ik}\,v_{ik}-w_{ii}\big)\,\partial_{v_{ik}}+\sum_{i<\ell'\leqslant \ell}\big(2\,v_{\ell' k}\,v_{ik}-w_{\ell' i}+w_{i\ell'}\big)\,\partial_{v_{\ell' k}}$$

$$+\sum_{1\leqslant r<s<i}\Big(v_{rk}\,\big[-w_{si}+w_{is}\big]+v_{sk}\,\big[w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{rs}}+\sum_{1\leqslant r<i}\Big(v_{rk}\,\big[w_{ii}\big]+v_{ik}\,\big[w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{ri}}$$
$$+\sum_{1\leqslant r\leqslant i}\Big(v_{rk}\,\big[w_{is}+w_{si}\big]+v_{sk}\,\big[w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{rs}}+\sum_{i<r\leqslant \ell}\Big(v_{ik}\,\big[w_{is}+w_{si}\big]+v_{sk}\,\big[-w_{ii}\big]\Big)\,\partial_{w_{is}}$$
$$+\sum_{i<r<s\leqslant \ell}\Big(v_{rk}\,\big[w_{is}+w_{si}\big]+v_{sk}\,\big[-w_{ri}-w_{ir}\big]\Big)\,\partial_{w_{rs}}$$

$$+\sum_{1\leqslant r<i}\Big(v_{rk}\,\big[-2\,w_{ri}+2\,w_{ir}\big]\Big)\,\partial_{w_{rr}}+\Big(v_{ik}\,\big[2\,w_{ii}\big]\Big)\,\partial_{w_{ii}}+\sum_{i<r\leqslant \ell}\Big(v_{rk}\,\big[2\,w_{ir}+2\,w_{ri}\big]\Big)\,\partial_{w_{rr}}$$

$$+\sum_{1\leqslant s<r<i}\Big(v_{rk}\,\big[-w_{si}+w_{is}\big]+v_{sk}\,\big[-w_{ri}+w_{ir}\big]\Big)\,\partial_{w_{rs}}+\sum_{1\leqslant s<i}\Big(v_{ik}\,\big[-w_{si}+w_{is}\big]+w_{sk}\,\big[w_{ii}\big]\Big)\,\partial_{w_{is}}$$
$$+\sum_{i<r\leqslant \ell}\Big(v_{rk}\,\big[-w_{si}+w_{is}\big]+v_{sk}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}+\sum_{i<r\leqslant \ell}\Big(v_{rk}\,\big[w_{ii}\big]+v_{ik}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{ri}}$$
$$+\sum_{i<s<r\leqslant \ell}\Big(v_{rk}\,\big[w_{is}+w_{si}\big]+v_{sk}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}.$$

For $1 \leqslant i \leqslant \ell$ and $1 \leqslant k \leqslant m$:

$$IL_{v_{ik}v_{ik}} := \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{n'=1}^{n} \big(2\, z_{in'}\, v_{\ell' k}\big)\, \partial_{z_{\ell' n'}}$$

$$+ \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{m'=1}^{m} \big(2\, v_{im'}\, v_{\ell' m'}\big)\, \partial_{v_{\ell' m'}}$$

$$+ \sqrt{-1} \sum_{\ell'=1}^{\ell} \sum_{\substack{m'=1 \\ m' \neq k}}^{m} \big(2\, v_{\ell' k}\, v_{im'}\big)\, \partial_{v_{\ell' m'}}$$

$$\overline{1} \sum_{1 \leqslant \ell' < i} \big(2\, v_{\ell' k}\, v_{ik} + w_{\ell' i} + w_{i \ell'}\big)\, \partial_{v_{\ell' k}} + \sqrt{-1}\, \big(2\, v_{ik}\, v_{ik} + w_{ii}\big)\, \partial_{v_{ik}} + \sqrt{-1} \sum_{i < \ell' \leqslant \ell} \big(2\, v_{\ell' k}\, v_{ik} + w_{\ell' i} - w_{i \ell'}\big)\, \partial_{v_{\ell' k}}$$

$$+ \sqrt{-1} \sum_{1 \leqslant r < s < i} \Big(v_{rk}\, \big[- w_{si} + w_{is}\big] + v_{sk}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sqrt{-1} \sum_{1 \leqslant r < i} \Big(v_{rk}\, \big[w_{ii}\big] + v_{ik}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{ri}}$$

$$+ \sqrt{-1} \sum_{1 \leqslant r \leqslant i} \Big(v_{rk}\, \big[w_{is} + w_{si}\big] + v_{sk}\, \big[w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}} + \sqrt{-1} \sum_{i < r \leqslant \ell} \Big(v_{ik}\, \big[w_{is} + w_{si}\big] + v_{sk}\, \big[- w_{ii}\big]\Big)\, \partial_{w_{is}}$$

$$+ \sqrt{-1} \sum_{i < r < s \leqslant \ell} \Big(v_{rk}\, \big[w_{is} + w_{si}\big] + v_{sk}\, \big[- w_{ri} - w_{ir}\big]\Big)\, \partial_{w_{rs}}$$

$$+ \sqrt{-1} \sum_{1 \leqslant r < i} \Big(v_{rk}\, \big[- 2\, w_{ri} + 2\, w_{ir}\big]\Big)\, \partial_{w_{rr}} + \Big(v_{ik}\, \big[2\, w_{ii}\big]\Big)\, \partial_{w_{ii}} + \sqrt{-1} \sum_{i < r \leqslant \ell} \Big(v_{rk}\, \big[2\, w_{ir} + 2\, w_{ri}\big]\Big)\, \partial_{w_{rr}}$$

$$+\sqrt{-1}\sum_{1\leqslant s<r<i}\Big(v_{rk}\,\big[-w_{si}+w_{is}\big]+v_{sk}\,\big[-w_{ri}+w_{ir}\big]\Big)\,\partial_{w_{rs}}$$
$$+\sqrt{-1}\sum_{1\leqslant s<i}\Big(v_{ik}\,\big[-w_{si}+w_{is}\big]+w_{sk}\,\big[w_{ii}\big]\Big)\,\partial_{w_{is}}$$
$$+\sqrt{-1}\sum_{i<r\leqslant\ell}\Big(v_{rk}\,\big[-w_{si}+w_{is}\big]+v_{sk}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}+\sqrt{-1}\sum_{i<r\leqslant\ell}\Big(v_{rk}\,\big[w_{ii}\big]+v_{ik}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{ri}}$$
$$+\sqrt{-1}\sum_{i<s<r\leqslant\ell}\Big(v_{rk}\,\big[w_{is}+w_{si}\big]+v_{sk}\,\big[w_{ir}+w_{ri}\big]\Big)\,\partial_{w_{rs}}.$$

4. $\mathfrak{su}(p,q)$ CR Models : Generators of $\mathfrak{g}_2$

There are 3 families of generators for $\mathfrak{g}_2$:

$$L_{w_{ij}w_{ij}} \qquad (1\leqslant i<j\leqslant\ell),$$
$$L_{w_{ii}w_{ii}} \qquad (1\leqslant i\leqslant\ell),$$
$$IL_{w_{ij}w_{ij}} \qquad (1\leqslant i>j\leqslant\ell).$$

For $1 \leqslant i < j \leqslant \ell$:

$$
\begin{aligned}
L_{w_{ij}w_{ij}} := & \sum_{1 \leqslant \ell' < i} \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{j\ell'} + w_{\ell' j} \big] + z_{jn'} \big[- w_{i\ell'} - w_{\ell' i} \big] \Big) \, \partial_{z_{\ell' n'}} \\
& + \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{ij} + w_{ji} \big] + z_{jn'} \big[- w_{ii} \big] \Big) \, \partial_{z_{in'}} \\
& + \sum_{i < \ell' < j} \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{j\ell'} + w_{\ell' j} \big] + z_{jn'} \big[w_{i\ell'} - w_{\ell' i} \big] \Big) \, \partial_{z_{\ell' n'}} \\
& + \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{jj} \big] + z_{jn'} \big[w_{ij} - w_{ji} \big] \Big) \, \partial_{z_{jn'}} \\
& + \sum_{j < \ell' \leqslant \ell} \sum_{n'=1}^{n} \Big(z_{in'} \big[- w_{j\ell'} + w_{\ell' j} \big] + z_{jn'} \big[w_{i\ell'} - w_{\ell' i} \big] \Big) \, \partial_{z_{\ell' n'}}
\end{aligned}
$$

$$+\sum_{1\leqslant \ell'<i}\sum_{m'=1}^{m}\Big(u_{im'}\big[w_{j\ell'}+w_{\ell' j}\big]+u_{jm'}\big[-w_{i\ell'}-w_{\ell' i}\big]\Big)\,\partial_{u_{\ell' m'}}$$

$$+\sum_{m'=1}^{m}\Big(u_{im'}\big[w_{ij}+w_{ji}\big]+u_{jm'}\big[-w_{ii}\big]\Big)\,\partial_{u_{im'}}$$

$$+\sum_{i<\ell'<j}\sum_{m'=1}^{m}\Big(u_{im'}\big[w_{j\ell'}+w_{\ell' j}\big]+u_{jm'}\big[w_{i\ell'}-w_{\ell' i}\big]\Big)\,\partial_{u_{\ell' m'}}$$

$$+\sum_{m'=1}^{m}\Big(u_{im'}\big[w_{jj}\big]+u_{jm'}\big[w_{ij}-w_{ji}\big]\Big)\,\partial_{u_{jm'}}$$

$$+\sum_{j<\ell'\leqslant \ell}\sum_{m'=1}^{m}\Big(u_{im'}\big[-w_{j\ell'}+w_{\ell' j}\big]+u_{jm'}\big[w_{i\ell'}-w_{\ell' i}\big]\Big)\,\partial_{u_{\ell' m'}}$$

$$+\sum_{1\leqslant\ell'<i}\sum_{m'=1}^{m}\Big(v_{im'}\big[w_{j\ell'}+w_{\ell'j}\big]+v_{jm'}\big[-w_{i\ell'}-w_{\ell'i}\big]\Big)\,\partial_{v_{\ell'm'}}$$

$$+\sum_{m'=1}^{m}\Big(v_{im'}\big[w_{ij}+w_{ji}\big]+v_{jm'}\big[-w_{ii}\big]\Big)\,\partial_{v_{im'}}$$

$$+\sum_{i<\ell'<j}\sum_{m'=1}^{m}\Big(v_{im'}\big[w_{j\ell'}+w_{\ell'j}\big]+v_{jm'}\big[w_{i\ell'}-w_{\ell'i}\big]\Big)\,\partial_{v_{\ell'm'}}$$

$$+\sum_{m'=1}^{m}\Big(v_{im'}\big[w_{jj}\big]+v_{jm'}\big[w_{ij}-w_{ji}\big]\Big)\,\partial_{v_{jm'}}$$

$$+\sum_{j<\ell'\leqslant\ell}\sum_{m'=1}^{m}\Big(v_{im'}\big[-w_{j\ell'}+w_{\ell'j}\big]+v_{jm'}\big[w_{i\ell'}-w_{\ell'i}\big]\Big)\,\partial_{v_{\ell'm'}}$$

$$
\begin{aligned}
&+ \sum_{1 \leqslant r < i} \big(-2\, w_{ri}\, w_{jr} + 2\, w_{rj}\, w_{ir} \big)\, \partial_{w_{rr}} \\
&\quad + \sum_{1 \leqslant r < i} \big(w_{ri}\, w_{ij} - w_{ir}\, w_{ji} + w_{ii}\, w_{jr} \big)\, \partial_{w_{ri}} \\
&\quad + \sum_{1 \leqslant r < i} \big(w_{rj}\, w_{ij} - w_{ir}\, w_{jj} + w_{ji}\, w_{jr} \big)\, \partial_{w_{rj}} \\
&\quad + \sum_{1 \leqslant s < i} \big(-w_{si}\, w_{ji} + w_{sj}\, w_{ii} + w_{is}\, w_{ij} \big)\, \partial_{w_{is}} \\
&\quad + \big(2\, w_{ii}\, w_{ij} \big)\, \partial_{w_{ii}} \\
&\quad + \sum_{i < s < j} \big(-w_{ii}\, w_{js} + w_{is}\, w_{ij} + w_{si}\, w_{ji} \big)\, \partial_{w_{is}} \\
&\quad + \big(-w_{ii}\, w_{jj} + w_{ij}^2 + w_{ji}^2 \big)\, \partial_{w_{ij}}
\end{aligned}
$$

$$+\sum_{j<s\leqslant\ell}\big(-w_{ii}\,w_{sj}+w_{ij}\,w_{is}+w_{ji}\,w_{si}\big)\,\partial_{w_{is}}$$

$$+\sum_{i<r<j}\big(w_{ii}\,w_{rj}+w_{ji}\,w_{ir}+w_{ij}\,w_{ri}\big)\,\partial_{w_{ri}}$$

$$+\sum_{i<r<j}\big(2\,w_{ir}\,w_{jr}+2\,w_{ri}\,w_{rj}\big)\,\partial_{w_{rr}}$$

$$+\sum_{i<r<j}\big(w_{ij}\,w_{rj}-w_{ri}\,w_{jj}+w_{ji}\,w_{jr}\big)\,\partial_{w_{rj}}$$

$$+\sum_{1\leqslant s<i}\big(-w_{si}\,w_{jj}+w_{sj}\,w_{ji}+w_{ij}\,w_{js}\big)\,\partial_{w_{js}}$$

$$+\big(2\,w_{ij}\,w_{ji}\big)\,\partial_{w_{ji}}$$

$$+\sum_{i<s<j}\big(w_{is}\,w_{jj}+w_{ij}\,w_{js}+w_{sj}\,w_{ji}\big)\,\partial_{w_{js}}$$

$$+\big(2\,w_{ij}\,w_{jj}\big)\,\partial_{w_{jj}}$$

$$+\sum_{j<s\leqslant\ell}\big(w_{ij}\,w_{js}-w_{ji}\,w_{sj}+w_{jj}\,w_{si}\big)\,\partial_{w_{js}}$$

$$+\sum_{j<r\leqslant\ell}\big(-w_{ii}\,w_{jr}+w_{ij}\,w_{ri}+w_{ir}\,w_{ji}\big)\,\partial_{w_{ri}}$$

$$+\sum_{j<r\leqslant\ell}\big(w_{ij}\,w_{rj}+w_{ir}\,w_{jj}-w_{ji}\,w_{jr}\big)\,\partial_{w_{rj}}$$

$$+\sum_{j<r\leqslant\ell}\big(2\,w_{ir}\,w_{rj}-2\,w_{jr}\,w_{ri}\big)\,\partial_{w_{rr}}$$

$$+\sum_{1\leqslant s<r<i}\Big(-w_{si}\,w_{jr}+w_{sj}\,w_{ir}-w_{ri}\,w_{js}+w_{rj}\,w_{is}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{1\leqslant r<s<i}\Big(w_{ri}\,w_{sj}-w_{rj}\,w_{si}-w_{ir}\,w_{js}+w_{is}\,w_{jr}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{1\leqslant r<i}\ \sum_{i<s<j}\Big(w_{ri}\,w_{sj}+w_{rj}\,w_{is}-w_{ir}\,w_{js}+w_{si}\,w_{jr}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{1\leqslant r<i}\ \sum_{j<s\leqslant\ell}\Big(-w_{ri}\,w_{js}+w_{rj}\,w_{is}-w_{ir}\,w_{sj}+w_{jr}\,w_{si}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{i<r<j}\ \sum_{1\leqslant s<i}\Big(-w_{si}\,w_{jr}+w_{sj}\,w_{ri}+w_{is}\,w_{rj}+w_{ir}\,w_{js}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{i<s<r<j}\Big(w_{is}\,w_{jr}+w_{ir}\,w_{js}+w_{si}\,w_{rj}+w_{sj}\,w_{ri}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{i<r<s<j}\Big(-w_{ir}\,w_{sj}+w_{is}\,w_{rj}-w_{ri}\,w_{js}+w_{si}\,w_{jr}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{i<r<j}\ \sum_{j<s\leqslant\ell}\Big(w_{ir}\,w_{js}+w_{is}\,w_{rj}-w_{ri}\,w_{sj}+w_{jr}\,w_{si}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{j<r\leqslant\ell}\ \sum_{1\leqslant s<i}\Big(-w_{si}\,w_{rj}+w_{sj}\,w_{ri}-w_{is}\,w_{jr}+w_{ir}\,w_{js}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{j<r\leqslant\ell}\ \sum_{i<s<j}\Big(w_{is}\,w_{rj}+w_{ir}\,w_{js}-w_{si}\,w_{jr}+w_{sj}\,w_{ri}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{j<s<r\leqslant\ell}\Big(w_{is}\,w_{rj}+w_{ir}\,w_{sj}-w_{js}\,w_{ri}-w_{jr}\,w_{si}\Big)\,\partial_{w_{rs}}$$

$$+\sum_{j<r<s\leqslant\ell}\Big(w_{ir}\,w_{js}-w_{is}\,w_{jr}-w_{ri}\,w_{sj}+w_{rj}\,w_{si}\Big)\,\partial_{w_{rs}}.$$

For $1\leqslant i\leqslant\ell$:

$$IL_{w_{ii}w_{ii}}\ :=\ \sqrt{-1}\sum_{1\leqslant\ell'<i}\ \sum_{n'=1}^{n} z_{in'}\,\big(w_{\ell'i}+w_{i\ell'}\big)\,\partial_{z_{\ell'n'}}$$

$$+\sqrt{-1}\sum_{n'=1}^{n} z_{in'}\,w_{ii}\,\partial_{z_{in'}}$$

$$\sqrt{-1}\sum_{i<\ell'\leqslant\ell}\ \sum_{n'=1}^{n} z_{in'}\,\big(w_{\ell'i}-w_{i\ell'}\big)\,\partial_{z_{\ell'n'}}$$

$$+\sqrt{-1}\sum_{1\leqslant\ell'<i}\sum_{m'=1}^{m}u_{im'}\left(w_{\ell' i}+w_{i\ell'}\right)\partial_{u_{\ell' m'}}$$

$$+\sqrt{-1}\sum_{m'=1}^{m}u_{im'}\,w_{ii}\,\partial_{u_{im'}}$$

$$+\sqrt{-1}\sum_{i<\ell'\leqslant\ell}\sum_{m'=1}^{m}u_{im'}\left(w_{\ell' i}-w_{i\ell'}\right)\partial_{u_{\ell' m'}}$$

$$+\sqrt{-1}\sum_{1\leqslant\ell'<i}\sum_{m'=1}^{m}v_{im'}\left(w_{\ell' i}+w_{i\ell'}\right)\partial_{v_{\ell' m'}}$$

$$+\sqrt{-1}\sum_{m'=1}^{m}v_{im'}\,w_{ii}\,\partial_{v_{im'}}$$

$$+\sqrt{-1}\sum_{i<\ell'\leqslant\ell}\sum_{m'=1}^{m}v_{im'}\left(w_{\ell' i}-w_{i\ell'}\right)\partial_{v_{\ell' m'}}$$

$$
\begin{aligned}
&+ \sqrt{-1} \sum_{1 \leqslant r < i} \big(- w_{ri}^2 + w_{ir}^2 \big)\, \partial_{w_{rr}} \\
&+ \sqrt{-1}\, w_{ii}\, w_{ii}\, \partial_{w_{ii}} \\
&+ \sqrt{-1} \sum_{i < r \leqslant \ell} \big(w_{ri}^2 - w_{ir}^2 \big)\, \partial_{w_{rr}} \\
&+ \sqrt{-1} \sum_{1 \leqslant s < i} w_{is}\, w_{ii}\, \partial_{w_{is}} \\
&+ \sqrt{-1} \sum_{i < s \leqslant \ell} w_{is}\, w_{ii}\, \partial_{w_{is}} \\
&+ \sqrt{-1} \sum_{1 \leqslant r < i} w_{ri}\, w_{ii}\, \partial_{w_{ri}} \\
&+ \sqrt{-1} \sum_{i < r \leqslant \ell} w_{ri}\, w_{ii}\, \partial_{w_{ri}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sqrt{-1} \sum_{1\leqslant r<s<i} \big(w_{ri}\, w_{is} - w_{si}\, w_{ir}\big)\, \partial_{w_{rs}} \\
&+ \sqrt{-1} \sum_{1\leqslant s<r<i} \big(- w_{si}\, w_{ri} + w_{is}\, w_{ir}\big)\, \partial_{w_{rs}} \\
&+ \sqrt{-1} \sum_{i<r<s\leqslant\ell} \big(- w_{ir}\, w_{si} + w_{is}\, w_{ri}\big)\, \partial_{w_{rs}} \\
&+ \sqrt{-1} \sum_{i<s<r\leqslant\ell} \big(- w_{is}\, w_{ir} + w_{si}\, w_{ri}\big)\, \partial_{w_{rs}} \\
&+ \sqrt{-1} \sum_{1\leqslant r<i} \sum_{i<s\leqslant\ell} \big(w_{ri}\, w_{si} + w_{ir}\, w_{is}\big)\, \partial_{w_{rs}} \\
&+ \sum_{i<r\leqslant\ell} \sum_{1\leqslant s<i} \big(w_{si}\, w_{ir} + w_{is}\, w_{ri}\big)\, \partial_{w_{rs}}.
\end{aligned}
$$

For $1 \leqslant j < i \leqslant \ell$:

$$
\begin{aligned}
IL_{w_{ij} w_{ij}} :=\ & \sqrt{-1} \sum_{1 \leqslant \ell' < j} \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{\ell' j} + w_{j \ell'} \big] + z_{jn'} \big[w_{\ell' i} + w_{i \ell'} \big] \Big) \, \partial_{z_{\ell' n'}} \\
& + \sqrt{-1} \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{jj} \big] + z_{jn'} \big[w_{ji} + w_{ij} \big] \Big) \, \partial_{z_{jn'}} \\
& + \sqrt{-1} \sum_{j < \ell' < i} \sum_{n'=1}^{n} \Big(z_{in'} \big[- w_{j \ell'} + w_{\ell' j} \big] + z_{jn'} \big[w_{\ell' i} + w_{i \ell'} \big] \Big) \, \partial_{z_{\ell' n'}} \\
& + \sqrt{-1} \sum_{n'=1}^{n} \Big(z_{in'} \big[w_{ij} - w_{ji} \big] + z_{jn'} \big[w_{ii} \big] \Big) \, \partial_{z_{in'}} \\
& + \sqrt{-1} \sum_{i < \ell' \leqslant \ell} \sum_{n'=1}^{n} \Big(z_{in'} \big[- w_{j \ell'} + w_{\ell' j} \big] + z_{jn'} \big[- w_{i \ell'} + w_{\ell' i} \big] \Big) \, \partial_{z_{\ell' n'}}
\end{aligned}
$$

$$+\sqrt{-1}\sum_{1\leqslant \ell'<j}\sum_{m'=1}^{m}\Big(u_{in'}\left[w_{\ell' j}+w_{j\ell'}\right]+u_{jm'}\left[w_{\ell' i}+w_{i\ell'}\right]\Big)\,\partial_{u_{\ell' m'}}$$

$$+\sqrt{-1}\sum_{m'=1}^{m}\Big(u_{im'}\left[w_{jj}\right]+u_{jm'}\left[w_{ji}+w_{ij}\right]\Big)\,\partial_{u_{jm'}}$$

$$+\sqrt{-1}\sum_{j<\ell'<i}\sum_{m'=1}^{m}\Big(u_{im'}\left[-w_{j\ell'}+w_{\ell' j}\right]+u_{jm'}\left[w_{\ell' i}+w_{i\ell'}\right]\Big)\,\partial_{u_{\ell' m'}}$$

$$+\sqrt{-1}\sum_{m'=1}^{m}\Big(u_{im'}\left[w_{ij}-w_{ji}\right]+u_{jm'}\left[w_{ii}\right]\Big)\,\partial_{u_{im'}}$$

$$+\sqrt{-1}\sum_{i<\ell'\leqslant \ell}\sum_{m'=1}^{m}\Big(u_{im'}\left[-w_{j\ell'}+w_{\ell' j}\right]+u_{jm'}\left[-w_{i\ell'}+w_{\ell' i}\right]\Big)\,\partial_{u_{\ell' m'}}$$

$$+\sqrt{-1}\sum_{1\leqslant\ell'<j}\ \sum_{m'=1}^{m}\Big(v_{in'}\,\big[w_{\ell'j}+w_{j\ell'}\big]+v_{jm'}\,\big[w_{\ell'i}+w_{i\ell'}\big]\Big)\,\partial_{v_{\ell'm'}}$$

$$+\sqrt{-1}\sum_{m'=1}^{m}\Big(v_{im'}\,\big[w_{jj}\big]+v_{jm'}\,\big[w_{ji}+w_{ij}\big]\Big)\,\partial_{v_{jm'}}$$

$$+\sqrt{-1}\sum_{j<\ell'<i}\ \sum_{m'=1}^{m}\Big(v_{im'}\,\big[-w_{j\ell'}+w_{\ell'j}\big]+v_{jm'}\,\big[w_{\ell'i}+w_{i\ell'}\big]\Big)\,\partial_{v_{\ell'm'}}$$

$$+\sqrt{-1}\sum_{m'=1}^{m}\Big(v_{im'}\,\big[w_{ij}-w_{ji}\big]+v_{jm'}\big[w_{ii}\big]\Big)\,\partial_{v_{im'}}$$

$$+\sqrt{-1}\sum_{i<\ell'\leqslant\ell}\ \sum_{m'=1}^{m}\Big(v_{im'}\,\big[-w_{j\ell'}+w_{\ell'j}\big]+v_{jm'}\,\big[-w_{i\ell'}+w_{\ell'i}\big]\Big)\,\partial_{v_{\ell'm'}}$$

$$+\sqrt{-1}\sum_{1\leqslant r<j}\big(-2\,w_{rj}\,w_{ri}+2\,w_{jr}\,w_{ir}\big)\,\partial_{w_{rr}}$$

$$+\sqrt{-1}\sum_{1\leqslant r<j}\big(w_{rj}\,w_{ij}+w_{ri}\,w_{jj}-w_{jr}\,w_{ji}\big)\,\partial_{w_{rj}}$$

$$+\sqrt{-1}\sum_{1\leqslant r<j}\big(w_{rj}\,w_{ii}+w_{ri}\,w_{ij}+w_{ji}\,w_{ir}\big)\,\partial_{w_{ri}}$$

$$+\sqrt{-1}\sum_{1\leqslant s<j}\big(-w_{sj}\,w_{ji}+w_{js}\,w_{ij}+w_{jj}\,w_{is}\big)\,\partial_{w_{js}}$$

$$+\sqrt{-1}\big(2\,w_{jj}\,w_{ji}\big)\,\partial_{w_{jj}}$$

$$+\sqrt{-1}\sum_{j<s<i}\big(-w_{jj}\,w_{si}+w_{js}\,w_{ij}+w_{ji}\,w_{sj}\big)\,\partial_{w_{js}}$$

$$+\sqrt{-1}\big(2\,w_{ij}\,w_{ji}\big)\,\partial_{w_{ji}}$$

$$+\sqrt{-1}\sum_{i<s\leqslant\ell}\big(w_{jj}\,w_{is}+w_{ji}\,w_{sj}+w_{js}\,w_{ij}\big)\,\partial_{w_{js}}$$

$$+\sqrt{-1}\sum_{j<r<i}\big(w_{jj}\,w_{ir}+w_{jr}\,w_{ji}+w_{rj}\,w_{ij}\big)\,\partial_{w_{rj}}$$

$$+\sqrt{-1}\sum_{j<r<i}\big(2\,w_{jr}\,w_{ri}+2\,w_{rj}\,w_{ir}\big)\,\partial_{w_{rr}}$$

$$+\sqrt{-1}\sum_{j<r<i}\big(-w_{jr}\,w_{ii}+w_{ji}\,w_{ir}+w_{ri}\,w_{ij}\big)\,\partial_{w_{ri}}$$

$$+\sqrt{-1}\sum_{1\leqslant s<j}\big(w_{si}\,w_{ji}+w_{js}\,w_{ii}+w_{is}\,w_{ij}\big)\,\partial_{w_{is}}$$

$$+\sqrt{-1}\big(w_{jj}\,w_{ii}+w_{ij}^2+w_{ji}^2\big)\,\partial_{w_{ij}}$$

$$+\sqrt{-1}\sum_{j<s<i}\big(w_{ji}\,w_{si}+w_{sj}\,w_{ii}+w_{ij}\,w_{is}\big)\,\partial_{w_{is}}$$

$$+\sqrt{-1}\big(2\,w_{ij}\,w_{ii}\big)\,\partial_{w_{ii}}$$

$$+\sqrt{-1}\sum_{i<s\leqslant\ell}\big(-w_{ji}\,w_{si}+w_{js}\,w_{ii}+w_{ij}\,w_{is}\big)\,\partial_{w_{is}}$$

$$+\sqrt{-1}\sum_{i<r\leqslant\ell}\big(w_{jj}\,w_{ri}+w_{ji}\,w_{jr}+w_{ij}\,w_{rj}\big)\,\partial_{w_{rj}}$$

$$+\sqrt{-1}\sum_{i<r\leqslant\ell}\big(-w_{ji}\,w_{ir}+w_{ij}\,w_{ri}+w_{ii}\,w_{rj}\big)\,\partial_{w_{ri}}$$

$$+\sqrt{-1}\sum_{i<r\leqslant\ell}\big(-2\,w_{jr}\,w_{ir}+2\,w_{rj}\,w_{ri}\big)\,\partial_{w_{rr}}$$

$$+\sqrt{-1}\sum_{1\leqslant s<r<j}\Big(-w_{sj}\,w_{ri}-w_{si}\,w_{rj}+w_{js}\,w_{ir}+w_{jr}\,w_{is}\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{1\leqslant r<s<j}\Big(w_{rj}\,w_{is}+w_{ri}\,w_{js}-w_{sj}\,w_{ir}-w_{si}\,w_{jr}\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{1\leqslant r<j}\;\sum_{j<s<i}\Big(w_{rj}\,w_{is}+w_{ri}\,w_{sj}-w_{jr}\,w_{si}+w_{js}\,w_{ir}\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{1\leqslant r<j}\;\sum_{i<s\leqslant\ell}\Big(w_{rj}\,w_{si}+w_{ri}\,w_{sj}+w_{is}\,w_{jr}+w_{js}\,w_{ir}\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{j<r<i}\;\sum_{1\leqslant s<j}\Big(-w_{sj}\,w_{ri}+w_{si}\,w_{jr}+w_{js}\,w_{ir}+w_{rj}\,w_{is}\Big)\,\partial_{w_{rs}}$$

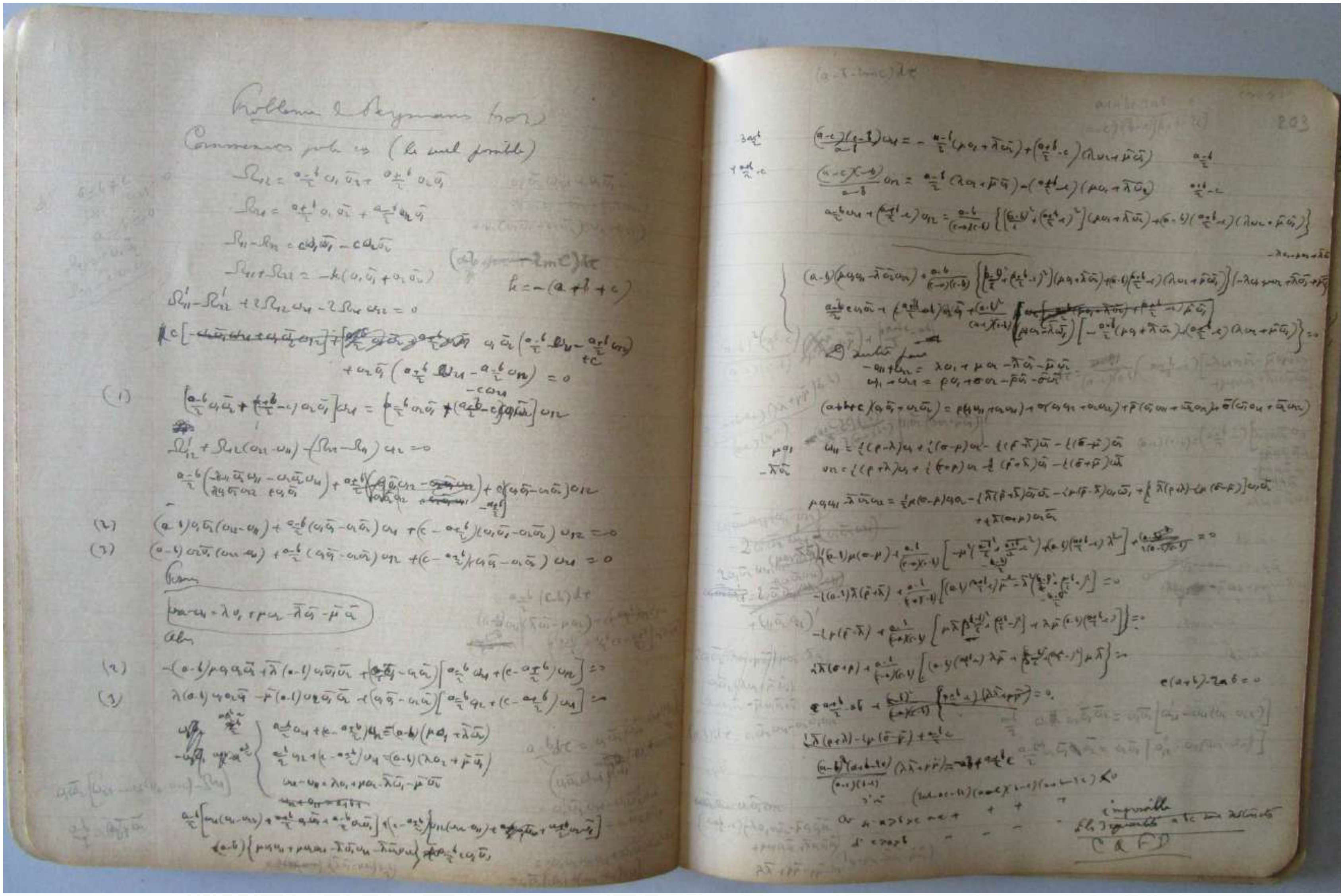
$$+\sqrt{-1}\sum_{j<s<r<i}\Big(w_{js}\,w_{ri}+w_{jr}\,w_{si}+w_{sj}\,w_{ir}+w_{rj}\,w_{is}\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{j<r<s<i}\Big(-w_{jr}\,w_{is}+w_{js}\,w_{ir}-w_{rj}\,w_{si}+w_{ri}\,w_{sj}\Big)\,\partial_{w_{rs}}$$

$$+\sqrt{-1}\sum_{j<r<i}\;\sum_{i<s\leqslant\ell}\Big(-w_{jr}\,w_{si}+w_{js}\,w_{ir}+w_{is}\,w_{rj}+w_{ri}\,w_{sj}\Big)\,\partial_{w_{rs}}$$

$$
\begin{aligned}
&+ \sqrt{-1} \sum_{i<r\leqslant \ell} \sum_{1\leqslant s<j} \Big(w_{sj}\, w_{ir} + w_{si}\, w_{jr} + w_{ri}\, w_{js} + w_{is}\, w_{rj}\Big)\, \partial_{w_{rs}} \\
&\quad + \sqrt{-1} \sum_{i<r\leqslant \ell} \sum_{j<s<i} \Big(- w_{js}\, w_{ir} + w_{jr}\, w_{si} + w_{sj}\, w_{ri} + w_{is}\, w_{rj}\Big)\, \partial_{w_{rs}} \\
&\quad + \sqrt{-1} \sum_{i<s<r\leqslant \ell} \Big(- w_{js}\, w_{ir} - w_{jr}\, w_{is} + w_{sj}\, w_{ri} + w_{si}\, w_{rj}\Big)\, \partial_{w_{rs}} \\
&\quad + \sqrt{-1} \sum_{i<r<s\leqslant \ell} \Big(- w_{jr}\, w_{si} + w_{js}\, w_{ri} - w_{ir}\, w_{sj} + w_{is}\, w_{rj}\Big)\, \partial_{w_{rs}}.
\end{aligned}
$$

Unpublished Manuscript of Élie Cartan 1938



Five-dimensional CR manifolds : Vector Field Method

- **Real hypersurface :**

$$M^5 \subset \mathbb{C}^3,$$

in coordinates :

$$(z, \zeta, w) = \big(z, \zeta, u + \sqrt{-1}\, v\big),$$

graphed as :

$$u = F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big).$$

- **Biholomorphic equivalences :**

$$(z, \zeta, w) \longmapsto \Big(f(z, \zeta, w),\, g(z, \zeta, w),\, h(z, \zeta, w)\Big) =: \big(z', \zeta', w'\big),$$

with :

$$0 \neq \begin{vmatrix} f_z & g_z & h_z \\ f_\zeta & g_\zeta & h_\zeta \\ f_w & g_w & h_w \end{vmatrix}.$$

- **Complex equation :**

$$w = Q\big(z, \zeta, \overline{z}, \overline{\zeta}, \overline{w}\big),$$

solve for w in :

$$\frac{w+\overline{w}}{2} = F\bigg(z, \zeta, \overline{z}, \overline{\zeta}, \frac{w-\overline{w}}{2i}\bigg)$$

using the implicit function theorem.

- **Insert map :**

$$(z, \zeta, w) \longmapsto \Big(f(z,\zeta,w),\, g(z,\zeta,w),\, h(z,\zeta,w)\Big) =: \big(z', \zeta', w'\big),$$

and get the *Fundamental Identity :*

$$h(z,\zeta,w) = Q'\Big(f(z,\zeta,w),\, g(z,\zeta,w),\, \overline{f}(\overline{z},\overline{\zeta},\overline{w}),\, \overline{g}(\overline{z},\overline{\zeta},\overline{w}),\, \overline{h}(\overline{z},\overline{\zeta},\overline{w})\Big)\bigg|_{w=Q(z,\zeta,\overline{z},\overline{\zeta},\overline{w})}.$$

- **Two Levi determinants :**

$$\begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{\zeta}} & Q_{\zeta\overline{w}} \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} Q'_{\overline{z}'} & Q'_{\overline{\zeta}'} & Q'_{\overline{w}'} \\ Q'_{z'\overline{z}'} & Q'_{z'\overline{\zeta}'} & Q'_{z'\overline{w}'} \\ Q'_{\zeta'\overline{z}'} & Q'_{\zeta'\overline{\zeta}'} & Q'_{\zeta'\overline{w}'} \end{vmatrix}.$$

• **Abbreviate :**

$$\mathscr{L}_z := \frac{\partial}{\partial z} + Q_z(z,\zeta,\overline{z},\overline{\zeta},\overline{w})\,\frac{\partial}{\partial w} \quad \text{and} \quad \mathscr{L}_\zeta := \frac{\partial}{\partial \zeta} + Q_\zeta(z,\zeta,\overline{z},\overline{\zeta},\overline{w})\,\frac{\partial}{\partial w}.$$

• **Proposition.** *Through any biholomorphism between real hypersurfaces* $\{w = Q\} \subset \mathbb{C}^3$ *and* $\{w' = Q'\} \subset \mathbb{C}'^3$*, one has :*

$$\begin{vmatrix} Q'_{\overline{z}'} & Q'_{\overline{\zeta}'} & Q'_{\overline{w}'} \\ Q'_{z'\overline{z}'} & Q'_{z'\overline{\zeta}'} & Q'_{z'\overline{w}'} \\ Q'_{\zeta'\overline{z}'} & Q'_{\zeta'\overline{\zeta}'} & Q'_{\zeta'\overline{w}'} \end{vmatrix} = \frac{\begin{vmatrix} f_z & f_\zeta & f_w \\ g_z & g_\zeta & g_w \\ h_z & h_\zeta & h_w \end{vmatrix}^3}{\begin{vmatrix} \overline{f}_{\overline{z}} & \overline{f}_{\overline{\zeta}} & \overline{f}_{\overline{w}} \\ \overline{g}_{\overline{z}} & \overline{g}_{\overline{\zeta}} & \overline{g}_{\overline{w}} \\ \overline{h}_{\overline{z}} & \overline{h}_{\overline{\zeta}} & \overline{h}_{\overline{w}} \end{vmatrix}^1} \frac{1}{\begin{vmatrix} \mathscr{L}_z(f) & \mathscr{L}_\zeta(f) \\ \mathscr{L}_z(g) & \mathscr{L}_\zeta(g) \end{vmatrix}^4} \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{\zeta}} & Q_{\zeta\overline{w}} \end{vmatrix},$$

hence the rank of the Levi determinant is invariant. □

• **First studies :**

[Cartan]

[Hachtroudi]

[Tanaka]

[Chern-Moser]

- **Maximal model :**

$$\mathsf{Re}\, w = u = z_1\overline{z}_1 \pm z_2\overline{z}_2,$$

with **15**-dimensional symmetry group :

$$\mathsf{SU}(3,1) \qquad\qquad \text{or} \qquad\qquad \mathsf{SU}(2,2).$$

- **Winkelmann hypersurface :**

$$\mathsf{Re}\,\big(w + \overline{z}_1 z_2\big) = |z_1|^4,$$

or :

$$u = x\,y + x^4$$

with **8**-dimensional symmetry group.

- **Classification of multiply transitive $M^5 \subset \mathbb{C}^3$:**

  - ☐ [Loboda]
  - ☐ [Doubrov-Medvedev-The]

Real affine surface $F(x,y,u)=0$	Affine hom.?	Classification	CR syms of $F(\mathfrak{Re}(z_1),\mathfrak{Re}(z_2),\mathfrak{Re}(w))=0$ beyond $i\partial_{z_1}, i\partial_{z_2}, i\partial_w$
$u=xy+x^4$	✓	N.8	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $z_1\partial_{z_1}+3z_2\partial_{z_2}+4w\partial_w$, $\partial_{z_1}-6(z_1)^2\partial_{z_2}+(z_2-2(z_1)^3)\partial_w$, $iz_1\partial_{z_1}+i(z_2-2(z_1)^3)\partial_{z_2}+i(z_1z_2-\frac{(z_1)^4}{2})\partial_w$
$u=xy+x\ln(x)$	✓	N.7-2 $\mathfrak{sl}(2,\mathbb{R})\ltimes(V_2\oplus V_0)$ $\varphi^{(+1)}$	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $z_1\partial_{z_1}-\partial_{z_2}+w\partial_w$, $i\frac{(z_1)^2}{2}\partial_{z_1}+i(w-z_1)\partial_{z_2}+iwz_1\partial_w$
$u=Xy+X\ln(X)$, $X=\exp(2x)+1$	×	N.7-2 $\mathfrak{su}(2)\ltimes(V_2\oplus V_0)$ $\varphi^{(-1)}$	$\cosh(z_1)\partial_{z_1}-\left(\frac{1}{2}\exp(-z_1)w+\exp(z_1)\right)\partial_{z_2}+w\sinh(z_1)\partial_w$, $\exp(-z_1)\partial_{z_2}+2\cosh(z_1)\partial_w$, $i\left(\exp(-z_1)\partial_{z_2}-2\sinh(z_1)\partial_w\right)$, $i\sinh(z_1)\partial_{z_1}+i\left(\frac{1}{2}\exp(-z_1)w-\exp(z_1)\right)\partial_{z_2}+iw\cosh(z_1)\partial_w$
$u=xy+x^\alpha$ $(\alpha\in\mathbb{R}\backslash\{0,1,2,3,4\})$	✓	N.6-1 $a^2=\frac{1-\alpha}{\alpha-4}$ $\in\mathbb{R}\backslash\{-1,-\frac{1}{4},0,\frac{1}{2},2\}$	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $z_1\partial_{z_1}+(\alpha-1)z_2\partial_{z_2}+\alpha w\partial_w$
$u=xy+\ln(x)$	✓	N.6-1 $a^2=-\frac{1}{4}$	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $z_1\partial_{z_1}-z_2\partial_{z_2}+\partial_w$
$u=xy+x^2\ln(x)$	✓	N.6-1 $a^2=\frac{1}{2}$	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $z_1\partial_{z_1}+(z_2-z_1)\partial_{z_2}+2w\partial_w$
$u=xy+x^3\ln(x)$	×	N.6-1 $a^2=2$	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $z_1\partial_{z_1}+(2z_2-\frac{3}{2}(z_1)^2)\partial_{z_2}+(3w-\frac{1}{2}(z_1)^3)\partial_w$
$u=y\exp(x)+\exp(\alpha x)$ $(\alpha\in\mathbb{R}\backslash\{-1,0,1,2\};$ $\alpha\sim 1-\alpha)$	×	N.6-2 $b^2=a^2=\frac{-(2\alpha-1)^2}{(\alpha+1)(\alpha-2)}$ $\in\mathbb{R}\backslash([-4,0)\cup\{\frac{1}{2}\})$	$\partial_{z_1}+(\alpha-1)z_2\partial_{z_2}+\alpha w\partial_w$, $\exp(\frac{1}{2}z_1)\partial_w+\exp(-\frac{1}{2}z_1)\partial_{z_2}$, $i\exp(\frac{1}{2}z_1)\partial_w-i\exp(-\frac{1}{2}z_1)\partial_{z_2}$
$u\cos(x)+y\sin(x)=\exp(\beta x)$ $(\beta\in\mathbb{R};\ \beta\sim-\beta)$	×	N.6-2 $b^2=a^2=\frac{-4\beta^2}{\beta^2+9}\in(-4,0]$	$\partial_{z_1}-(\beta z_2+w)\partial_{z_2}+(z_2-\beta w)\partial_w$, $\sin(z_1)\partial_{z_2}-\cos(z_1)\partial_w$, $-i\cos(z_1)\partial_{z_2}-i\sin(z_1)\partial_w$
$u=xy+\exp(x)$	✓	N.6-2 $b^2=a^2=-4$	$\partial_{z_2}+z_1\partial_w$, $iz_1\partial_{z_2}+i\frac{(z_1)^2}{2}\partial_w$, $\partial_{z_1}+z_2\partial_{z_2}+(w+z_2)\partial_w$
$u=y\exp(x)-\frac{x^2}{2}$	×	N.6-2 $b^2=a^2=\frac{1}{2}$	$\partial_{z_1}-z_2\partial_{z_2}-z_1\partial_w$, $\exp(\frac{1}{2}z_1)\partial_w+\exp(-\frac{1}{2}z_1)\partial_{z_2}$, $i\exp(\frac{1}{2}z_1)\partial_w-i\exp(-\frac{1}{2}z_1)\partial_{z_2}$

TABLE 7. Real affine surfaces and symmetries of corresponding tubular CR structures: type N cases

- **Simply transitive $M^5 \subset \mathbb{C}^3$:**
  - ☐ [Loboda]
  - ☐ [Doubrov-Merker-The]
- **Tube hypersurfaces :**

$$M^5 = S^2 \times i\,\mathbb{R}^3$$
$$\mathsf{Re}\, w = F\big(\mathsf{Re}\, z_1,\ \mathsf{Re}\, z_2\big).$$

- **Biholomorphisms Lie algebra :**

$$L = A(z,\zeta,w)\,\frac{\partial}{\partial z} + B(z,\zeta,w)\,\frac{\partial}{\partial \zeta} + C(z,\zeta,w)\,\frac{\partial}{\partial w},$$

with conjugate :

$$\overline{L} = \overline{A}\big(\overline{z},\overline{\zeta},\overline{w}\big)\,\frac{\partial}{\partial \overline{z}} + \overline{B}\big(\overline{z},\overline{\zeta},\overline{w}\big)\,\frac{\partial}{\partial \overline{\zeta}} + \overline{C}\big(\overline{z},\overline{\zeta},\overline{w}\big)\,\frac{\partial}{\partial \overline{w}},$$

- **Infinitesimal symmetries :**

$$\mathfrak{hol}(M) := \big\{L : \ \big(L + \overline{L}\big)\big|_M \text{ is tangent to } M\big\}.$$

• **Observation.** *Affine symmetries yield holomorphic symmetries.*

$$S^2 \subset \mathbb{R}^3 \qquad\qquad u = x_1^2 + x_2^2,$$
$$M^5 \subset \mathbb{C}^3 \qquad\qquad \mathsf{Re}\, w = (\mathsf{Re}\, z_1)^2 + (\mathsf{Re}\, z_2)^2,$$

in coordinates :

$$z_1 = x_1 + i\, x_1, \qquad\qquad z_2 = x_2 + i\, x_2, \qquad\qquad w = u + i\, v.$$

• **4 affine generators :**

$$\begin{aligned} e_1 &:= x_1\, \partial_{x_1} + x_2\, \partial_{x_2}, \\ e_2 &:= x_2\, \partial_{x_1} - x_1\, \partial_{x_2}, \\ e_3 &:= \partial_{x_1} + 2\, x_1\, \partial_u, \\ e_4 &:= \partial_{x_2} + 2\, x_2\, \partial_u. \end{aligned}$$

• **4 holomorphic generators :** [Replace $x_1 \longmapsto z_1$]

$$\begin{aligned} e_1^c &:= z_1\, \partial_{z_1} + z_2\, \partial_{z_2}, \\ e_2^c &:= z_2\, \partial_{z_1} - z_1\, \partial_{z_2}, \\ e_3^c &:= \partial_{z_1} + 2\, z_1\, \partial_w, \\ e_4^c &:= \partial_{z_2} + 2\, z_2\, \partial_w. \end{aligned}$$

- **Corollary.** *Affine homogeneity implies holomorphic homogeneity :*

$$\mathfrak{hol}\big(S^2 \times i\,\mathbb{R}^3\big) \supset \mathfrak{aff}(S^2) \oplus i\,\mathbb{R}^3.$$

- **From the Doubrov-Komrakov-Rabinovich list, exclude :**

DKR label	Non-degenerate real affine surface	ILC Classification [7]
(3)	$u = \ln(x_1) + \alpha \ln(x_2)$ $(\alpha \neq 0)$	D.7: $\alpha \neq 0, -1$; O.15: $\alpha = -1$
(4)	$u = \alpha \arg(ix_1 + x_2) + \ln(x_1^2 + x_2^2)$ $u = \arg(ix_1 + x_2)$	D.7 O.15
(7)	$u = x_2^2 + \epsilon e^{x_1}$	O.15
(8)	$u = x_2^2 + \epsilon x_1^\alpha$ $(\alpha \neq 0, 1)$	D.6-2: $\alpha \neq 0, 1, 2$; O.15: $\alpha = 2$
(9)	$u = x_2^2 + \epsilon \ln(x_1)$	D.7
(10)	$u = x_2^2 + \epsilon x_1 \ln(x_1)$	D.6-2
(11)	$u = x_1 x_2 + e^{x_1}$	N.6-2
(12)	$u = x_1 x_2 + x_1^\alpha$	N.6-1: $\alpha \neq 0, 1, 2, 3, 4$; N.8: $\alpha = 4$; O.15: $\alpha = 0, 1, 2, 3$
(13)	$u = x_1 x_2 + \ln(x_1)$	N.6-1
(14)	$u = x_1 x_2 + x_1 \ln(x_1)$	N.7-2
(15)	$u = x_1 x_2 + x_1^2 \ln(x_1)$	N.6-1
(17)	$x_1 u = x_2^2 + \epsilon x_1 \ln(x_1)$	D.6-1

TABLE 2. Affinely simply-transitive surfaces with holomorphically multiply-transitive associated tubes. Parameters $\alpha \in \mathbb{R}$ and $\epsilon = \pm 1$.

• Simply transitive classification : [Doubrov-M.-The]

Label	Affinely simply-transitive non-degenerate real surface $\mathcal{F}(x_1, x_2, u) = 0$	Holomorphic symmetries of $\mathcal{F}(\mathrm{Re}(z_1), \mathrm{Re}(z_2), \mathrm{Re}(w)) = 0$ beyond $i\partial_{z_1}, i\partial_{z_2}, i\partial_w$	Levi-definite condition
T1	$u = x_1^\alpha x_2^\beta$ Non-degeneracy: $\alpha\beta(1-\alpha-\beta) \neq 0$ Restriction: $(\alpha,\beta) \neq (1,1), (-1,1), (1,-1)$ Redundancy: $(\alpha,\beta) \sim (\beta,\alpha) \sim \left(\frac{1}{\alpha}, -\frac{\beta}{\alpha}\right)$	$z_1\partial_{z_1} + \alpha w\partial_w$, $z_2\partial_{z_2} + \beta w\partial_w$	$\alpha\beta(1-\alpha-\beta) > 0$
T2	$u = \left(x_1^2 + x_2^2\right)^\alpha \exp\left(\beta \arctan(\frac{x_2}{x_1})\right)$ Non-degeneracy: $\alpha \neq \frac{1}{2}$ & $(\alpha,\beta) \neq (0,0)$ Restriction: $(\alpha,\beta) \neq (1,0)$ Redundancy: $(\alpha,\beta) \sim (\alpha,-\beta)$	$z_1\partial_{z_1} + z_2\partial_{z_2} + 2\alpha w\partial_w$, $z_2\partial_{z_1} - z_1\partial_{z_2} - \beta w\partial_w$	$\alpha > \frac{1}{2}$
T3	$u = x_1\left(\alpha \ln(x_1) + \ln(x_2)\right)$ Non-degeneracy: $\alpha \neq -1$	$z_1\partial_{z_1} - \alpha z_2\partial_{z_2} + w\partial_w$, $z_2\partial_{z_2} + z_1\partial_w$	$\alpha < -1$
T4	$\left(u - x_1x_2 + \frac{x_1^3}{3}\right)^2 = \alpha\left(x_2 - \frac{x_1^2}{2}\right)^3$ Non-degeneracy: $\alpha \neq -\frac{8}{9}$ Restriction: $\alpha \neq 0$	$z_1\partial_{z_1} + 2z_2\partial_{z_2} + 3w\partial_w$, $\partial_{z_1} + z_1\partial_{z_2} + z_2\partial_w$	$\alpha < -\frac{8}{9}$
T5	$x_1u = x_2^2 + \epsilon x_1^\alpha$ Non-degeneracy: $\alpha \neq 1, 2$ Restriction: $\alpha \neq 0$	$z_1\partial_{z_1} + \frac{\alpha}{2}z_2\partial_{z_2} + (\alpha-1)w\partial_w$, $z_1\partial_{z_2} + 2z_2\partial_w$	$\epsilon(\alpha-1)(\alpha-2) > 0$
T6	$x_1u = x_2^2 + \epsilon x_1^2 \ln(x_1)$	$z_1\partial_{z_1} + z_2\partial_{z_2} + (\epsilon z_1 + w)\partial_w$, $z_1\partial_{z_2} + 2z_2\partial_w$	$\epsilon = +1$

TABLE 1. All simply-transitive *tubes* $M^5 \subset \mathbb{C}^3$. Parameters $\alpha, \beta \in \mathbb{R}$ and $\epsilon = \pm 1$.

Theorem 1.1. *Any simply-transitive Levi non-degenerate hypersurface $M^5 \subset \mathbb{C}^3$ is locally biholomorphic to precisely one of the following.*

(1) *Either one hypersurface among the 6 families of tubular hypersurfaces listed in Table 1 below, with corresponding 5 generators of $\mathfrak{hol}(M)$.*

(2) *Or the single* nontubular *exceptional model:*

$$\mathrm{Im}(w) \,=\, \big|\,\mathrm{Im}(z_2) - w\,\mathrm{Im}(z_1)\big|^2, \tag{1.2}$$

having indefinite Levi signature and the infinitesimal symmetries:

$$z_1\,\partial_{z_1} - z_2\,\partial_{z_2} - 2w\,\partial_w, \qquad z_1\,\partial_{z_2} + \partial_w, \qquad z_2\,\partial_{z_1} - w^2\,\partial_w, \qquad \partial_{z_1}, \qquad \partial_{z_2}, \tag{1.3}$$

with Lie algebra structure $\mathfrak{saff}(2,\mathbb{R}) := \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, i.e. the planar equi-affine Lie algebra.

• Insure Hessian rank 2 :

Family	Equation	Parameters	Hessian	Rank two
$(\mathbf{1})$	$u = x^\alpha y^\beta$	$(\alpha,\beta) \in \mathbb{R}^2$	$\alpha\beta(1-\alpha-\beta)\,x^{2\alpha-2}y^{2\beta-2}$	$\alpha \neq 0,\ \beta \neq 0,\ \alpha+\beta \neq 1$
$(\mathbf{2})$	$u = (x^2+y^2)^\beta e^{\alpha \arctan \frac{y}{x}}$	$(\alpha,\beta) \in \mathbb{R}^2$	$(2\beta-1)(\alpha^2+4\beta^2)(x^2+y^2)^{2\beta-2}e^{2\alpha \arctan \frac{y}{x}}$	$\beta \neq \frac{1}{2},\ (\alpha,\beta) \neq (0,0)$
$(\mathbf{5})$	$u = x(\alpha \log x + \log y)$	$\alpha \in \mathbb{R}$	$\frac{-\alpha-1}{y^2}$	$\alpha \neq -1$
$(\mathbf{6})$	$\big(u - xy + \frac{x^3}{3}\big)^2 = \alpha\big(y - \frac{x^2}{2}\big)^3$	$\alpha \in \mathbb{R}$	$-1 - \frac{9}{8}\alpha$	$\alpha \neq -\frac{8}{9}$
$(\mathbf{16})$	$xu = y^2 \pm x^\alpha$	$\alpha \in \mathbb{R}$	$\pm 2(\alpha-1)(\alpha-2)x^{\alpha-4}$	$\alpha \neq 1,\ \alpha \neq 2$
$(\mathbf{18})$	$xu = y^2 \pm x^2 \log x$	$\emptyset$	$\pm\frac{2}{x^2}$	always

• Remaining simply transitive surfaces :

#	Equation	Hessian rank = 2	Genericity hypotheses	Affine symmetries	Lie brackets	Structure
$(\mathbf{1})$	$u = x^\alpha y^\beta$	$\alpha \neq 0$ $\beta \neq 0$ $\alpha + \beta \neq 1$	$(\alpha, \beta) \neq (1, 1)$ $\neq (-1, 1)$ $\neq (1, -1)$	$x\partial_x + \alpha u \partial_u$ $y\partial_y + \beta u \partial_u$	0	$\mathfrak{a}_2$
$(\mathbf{2})$	$u = (x^2 + y^2)^\beta e^{\alpha \arctan \frac{y}{x}}$	$\beta \neq \frac{1}{2}$ $(\alpha, \beta) \neq (0, 0)$	$(\alpha, \beta) \neq (0, 1)$	$x\partial_x + y\partial_y + 2\beta u \partial_u$ $-y\partial_x + x\partial_y + \alpha u \partial_u$	0	$\mathfrak{a}_2$
$(\mathbf{5})$	$u = x(\alpha \log x + \log y)$	$\alpha \neq -1$	$\emptyset$	$x\partial_x - \alpha y \partial_y + u\partial_u$ $y\partial_y + x\partial_u$	0	$\mathfrak{a}_2$
$(\mathbf{6})$	$\big(u - xy + \frac{x^3}{3}\big)^2 = \alpha\big(y - \frac{x^2}{2}\big)^3$	$\alpha \neq -\frac{8}{9}$	$\alpha \neq 0$	$x\partial_x + 2y\partial_y + 3u\partial_u$ $\partial_x + x\partial_y + y\partial_u$	$[e_1, e_2] = -e_2$	$\mathfrak{r}_2$
$(\mathbf{16})$	$xu = y^2 \pm x^\alpha$	$\alpha \neq 1$ $\alpha \neq 2$	$\alpha \neq 0$	$x\partial_x + \frac{\alpha}{2} y \partial_y + (\alpha - 1) u \partial_u$ $x\partial_y + 2y\partial_u$	$[e_1, e_2] = (1 - \frac{\alpha}{2}) e_2$	$\mathfrak{r}_2$
$(\mathbf{18})$	$xu = y^2 \pm x^2 \log x$	yes	none	$x\partial_x + y\partial_y + (u \pm x)\partial_u$ $x\partial_y + 2y\partial_u$	0	$\mathfrak{a}_2$

• Specify affine exceptions :

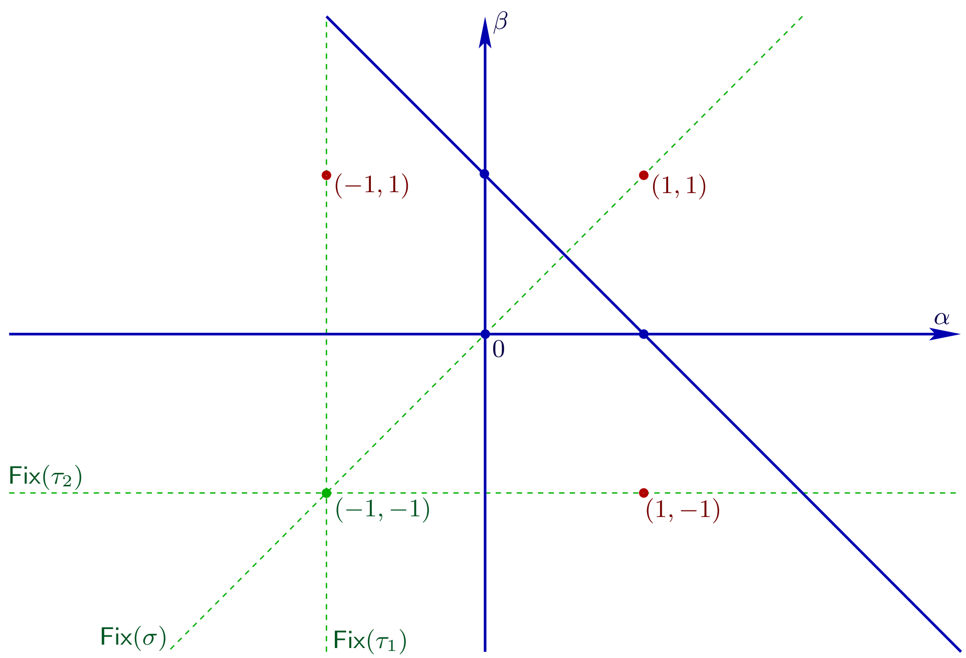
#	Equation	Affine symmetries	Lie brackets	Structure
$(\mathbf{1})$	$u = xy$	$-x\partial_x + y\partial_y$ $x\partial_x + u\partial_u$ $\partial_x + y\partial_u$ $\partial_y + x\partial_u$	$[e_1, e_3] = e_3$ $[e_1, e_4] = -e_4$ $[e_2, e_3] = -e_3$	$\mathfrak{r}_4$
$(\mathbf{2})$	$u = x^2 + y^2$	$-y\partial_x + x\partial_y$ $\partial_x + 2x\partial_u$ $\partial_y + 2y\partial_u$ $x\partial_x + y\partial_y + 2u\partial_u$	$[e_1, e_2] = -e_3$ $[e_1, e_3] = e_2$ $[e_2, e_4] = e_2$ $[e_3, e_4] = e_3$	$\mathfrak{r}_4$
$(\mathbf{6})$	$u = xy - \frac{x^3}{3}$	$\partial_y + x\partial_u$ $x\partial_x + 2y\partial_y + 3u\partial_u$ $\partial_x + x\partial_y + y\partial_u$	$[e_1, e_2] = 2e_1$ $[e_2, e_3] = -e_3$	$\mathfrak{r}_3$
$(\mathbf{16})$	$xu = y^2 \pm 1$	$2y\partial_x + u\partial_y$ $x\partial_x - u\partial_u$ $x\partial_y + 2y\partial_u$	$[e_1, e_2] = e_1$ $[e_1, e_3] = -2e_2$ $[e_2, e_3] = e_3$	$\mathfrak{sl}(2, \mathbb{R})$

For the family $\{u = x^\alpha y^\beta\}$, on the parameter space $\mathbb{R}^2 \ni (\alpha, \beta)$, introduce three discrete symmetries that are involutions :

$$\sigma(\alpha,\beta) := (\beta,\alpha), \qquad \tau_1(\alpha,\beta) := \big(\tfrac{1}{\alpha}, -\tfrac{\beta}{\alpha}\big), \qquad \tau_2(\alpha,\beta) := \big(-\tfrac{\alpha}{\beta}, \tfrac{1}{\beta}\big).$$

They account for the freedom of swapping variables names and of changing exponents :

$$u = x^\alpha y^\beta \longleftrightarrow u = y^\alpha x^\beta, \;\; u = x^\alpha y^\beta \longleftrightarrow u^{1/\alpha} y^{-\beta/\alpha} = x, \;\; u = x^\alpha y^\beta \longleftrightarrow u^{1/\beta} x^{-\alpha/\beta} = y.$$



In dotted lines, draw their respective fixed point sets :

$$\mathsf{Fix}(\sigma) = \{\beta = \alpha\}, \qquad \mathsf{Fix}(\tau_1) = \{\alpha = -1\}, \qquad \mathsf{Fix}(\tau_2) = \{\beta = -1\}.$$

So if S^2 is affinely transitive, then $M^5 = S^2 \times i\mathbb{R}^3$ is CR-transitive.

• **Question** *For which values of parameters do the tubifications of* **(1)**, **(2)**, **(6)**, **(16)**, **(18)** *remain simply transitive ?*

Several computations using different, doubly-checked methods show the

• **Proposition.** *After tubifying, exceptions to simple CR-transitivity of* $S^2 \times i\mathbb{R}^3$ *are the same as exceptions to simple affine transitivity of* S^2.

• **Question :** *What is the computational method ?*

"Imaginary" universes are so much more beautiful than this stupidly constructed "real" one. Godfrey Harold Hardy.

Infinitesimal Symmetries

By complexifying the defining equation of a real hypersurface $M^5 \subset \mathbb{C}^3$, one gets a complex hypersurface in $\mathbb{C}^3_{x,y,z} \times \mathbb{C}^3_{a,b,c}$ graphed as :

$$M: \qquad z = Q(x, y, a, b, c),$$

with Q analytic, *i.e.* expandable in converging power series. We may assume $0 \in M$, *i.e.* $0 = Q(0,0,0,0,0)$.

Consider the infinite-dimensional group of local analytic transformations :

$$(x, y, z,\, a, b, c) \longmapsto \big(x'(x,y,z), y'(x,y,z), z'(x,y,z),\, a'(a,b,c), b'(a,b,c), c'(a,b,c)\big),$$

not necessarily fixing the origin. Define $\mathsf{Sym}(M)$ to be those transformations which stabilize M, near the origin.

Infinitesimal generators are those vector fields :

$$L = X(x,y,z)\,\partial_x + Y(x,y,z)\,\partial_y + Z(x,y,z)\,\partial_z + A(a,b,c)\,\partial_a + B(a,b,c)\,\partial_b + C(a,b,c)\,\partial_c$$

that are tangent to M. Denote $\mathfrak{sym}(M)$ this collection.

It is known that M is Levi nondegenerate if and only if :

$$0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix}.$$

This condition is invariant. It suffices to assume it holds at the origin, which we will do from now on.

There are general arguments showing that $\mathsf{Sym}(M)$ is a finite-dimensional local Lie group, with $\mathsf{Lie}\,\mathsf{Sym}(M) = \mathfrak{sym}(M)$. However, these arguments, which work in any dimension and in any codimension, are rough concerning $\dim \mathfrak{sym}(M)$ and Lie algebraic data.

Our goal is to explain a "*primitive*" computational method, based on power series expansions of orders 0, 1, 2, 3, 4, 5, 6, not more, to determine $\mathfrak{sym}(M)$ for any given M, namely for any given Q.

We will make the assumption that M is *rigid :*

$$z = -c + F(x, y, a, b),$$

with $0 = F(0,0,0,0)$, namely $Q = -c + F$. This is justified when M is homogeneous (our concern), whence there exists at least one

$L \in \mathfrak{syn}(M)$ with $L(0)$ not tangent to the 4-dimensional contact distribution. After a straightening, one can make $L = \partial_z - \partial_c$, and tangency to $\{z = Q\}$ forces $Q = -c + F$ as above.

A subclass of rigid M consists of *tubes :*

$$z = -c + F(x + y,\, a + b).$$

The method will work (much) better after performing an easy *prenormalization*. Rewrite :

$$z - F(x, y, 0, 0) = -\big(c - F(0, 0, a, b)\big) + F(x, y, a, b) - F(x, y, 0, 0) - F(0, 0, a, b),$$

which becomes, afer an allowed transformation :

$$z = -c + F(x, y, a, b) \quad \text{with} \quad \begin{cases} 0 \equiv F(x, y, 0, 0), \\ 0 \equiv F(0, 0, a, b). \end{cases}$$

Specifying second order terms, we get :

$$z = -c + e\, xa + f\, xb + g\, ya + h\, yb + G(x, y, a, b) \quad \text{with} \quad \begin{cases} 0 \equiv G(x, y, 0, 0), \\ 0 \equiv G(0, 0, a, b), \end{cases}$$

with complex constants e, f, g, h satisfying $0 \neq \left|\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right|$, and with $G = \mathrm{O}_{x,y,a,b}(3)$.

Often, the graphing function depends on some (complex) parameters :

$$F = F_{\alpha,\beta,\gamma,\dots}(x,y,a,b).$$

In particular, e, f, g, h may depend on α, β, γ, $\dots$. The power series expansion of F may involve some complexity.

To diagonalize the Levi form, one could employ the allowed transformation :

$$a' := e\,a + f\,b, \qquad\qquad b' := g\,a + h\,b,$$

so that :

$$z = -c + x\,a' + y\,b' + G\Big(x, y, \tfrac{ha' - fb'}{eh - fg}, \tfrac{eb' - ga'}{eh - fg}\Big).$$

However, sometimes, doing this increases even more the complexity of power series coefficients, and Maple happens to be lost. Therefore, we will present the method without diagonalizing the Levi form.

Abbreviate the *Levi quadratic terms* as :

$$\ell(x,y,a,b) := e\,xa + f\,xb + g\,ya + h\,yb.$$

Now, the ***tangency condition :***

$$
\begin{aligned}
0 &\equiv L\big(-z-c+F(x,y,a,b)\big)\Big|_{z:=-c+F} \\
&\equiv \big[X\,F_x+Y\,F_y-Z+A\,F_a+B\,F_b-C\big]\Big|_{z=-c+F},
\end{aligned}
$$

reads as the identical vanishing of the following power series in 5 variables (x,y,a,b,c) :

$$
\begin{aligned}
0 \equiv\; & X\big(x,y,-c+F(x,y,a,b)\big)\,F_x(x,y,a,b) \\
& +Y\big(x,y,-c+F(x,y,a,b)\big)\,F_y(x,y,a,b)-Z\big(x,y,-c+F(x,y,a,b)\big) \\
& +A(a,b,c)\,F_a(x,y,a,b)+B(a,b,c)\,F_b(x,y,a,b)-C(a,b,c) \\
=:\; & \mathsf{eqdef}(F).
\end{aligned}
$$

Since $z=-c+\ell+G=-c+\mathrm{O}_{=2}+\mathrm{O}_{\geqslant 3}$, it is natural to assign weights :

$$
[x]:=1=:[y], \qquad [z]:=2, \qquad\qquad [a]:=1=:[b], \qquad [c]:=2,
$$

and to expand in weighted homogeneous degrees :

$$
\begin{aligned}
G &= \sum_{i,j,l,m,n} G_{i,j,l,m,n}\, x^i y^j a^l b^m c^n \\
&= \sum_{\mu\geqslant 3} \bigg(\sum_{i+j+l+m+2n=\mu} G_{i,j,l,m,n}\, x^i y^j a^l b^m c^n \bigg).
\end{aligned}
$$

Similarly [why there is a shift $\mu \longmapsto \mu - 1$ in X, Y will be understood later] :

$$
\begin{aligned}
X &= \sum_{i,j,k} X_{i,j,k}\, x^i y^j z^k = \sum_{\mu\geqslant 1} \bigg(\sum_{i+j+2k=\mu-1} X_{i,j,k}\, x^i y^j z^k \bigg), \\
Y &= \sum_{i,j,k} Y_{i,j,k}\, x^i y^j z^k = \sum_{\mu\geqslant 1} \bigg(\sum_{i+j+2k=\mu-1} Y_{i,j,k}\, x^i y^j z^k \bigg), \\
Z &= \sum_{i,j,k} Z_{i,j,k}\, x^i y^j z^k = \sum_{\mu\geqslant 0} \bigg(\sum_{i+j+2k=\mu} Z_{i,j,k}\, x^i y^j z^k \bigg),
\end{aligned}
$$

and quite similarly for A, B, C.

Introduce the *homogeneous variables*, for $\mu = 0$:

$$V_0 \ := \ \big\{ Z_{0,0,0},\ C_{0,0,0} \big\},$$

and for $\mu = 1, 2, \dots$:

$$V_\mu \ := \ \Big\{ \big(X_{i,j,k}\big)_{i+j+2k=\mu-1},\ \big(Y_{i,j,k}\big)_{i+j+2k=\mu-1},\ \big(Z_{i,j,k}\big)_{i+j+2k=\mu},$$
$$\big(A_{l,m,n}\big)_{l+m+2n=\mu-1},\ \big(B_{l,m,n}\big)_{l+m+2n=\mu-1},\ \big(C_{l,m,n}\big)_{l+m+2n=\mu} \Big\}.$$

Their number is :

$$\# V_\mu \ = \ 4\,\mathsf{Card}\,\big\{ i+j+2k = \mu-1 \big\} + 2\,\mathsf{Card}\,\big\{ i+j+2k = \mu \big\}.$$

By some elementary reasoning, it is known that every $L \in \mathfrak{sym}(M)$ is "*uniquely determined by its second order jet at the origin*" :

$$j^2 L \ := \ \big\{ X_{i,j,k},\ Y_{i,j,k},\ Z_{i,j,k},\ A_{l,m,n},\ B_{l,m,n}, C_{l,m,n} \big\}_{i+j+k \leqslant 2}^{l+m+n \leqslant 2},$$

which makes 60 complex constants, and more finely :

$$\mathsf{dim}\,\mathfrak{sym}(M) \ \leqslant \ 15 \ = \ \mathsf{dim}\,\mathfrak{sym}\,\big\{ z = -c + xa + yb \big\}.$$

Our goal is to explain this, and go beyond.

By the tangency condition, all coefficients of all monomials $x^iy^ja^lb^mc^n$ should vanish :

$$0 = [x^iy^ja^lb^mc^n]\big(\mathsf{eqdef}(\ell+G)\big) \qquad (\forall\, i,j,l,m,n).$$

Mainly, we will compare this with the quadric $\{z=-c+\ell\}$ when $G=$ remainder vanishes identically :

$$0 = [x^iy^ja^lb^mc^n]\big(\mathsf{eqdef}(\ell)\big) \qquad (\forall\, i,j,l,m,n).$$

Accordingly, denote :

$$\begin{aligned} E_{i,j,l,m,n}^{\ell} &:= [x^iy^ja^lb^mc^n]\big(\mathsf{eqdef}(\ell)\big), \\ E_{i,j,l,m,n}^{\ell+G} &:= [x^iy^ja^lb^mc^n]\big(\mathsf{eqdef}(\ell+G)\big). \end{aligned}$$

Let us now explain a key *phenomenon of triangularity* of these linear equations, whatever $G = \mathrm{O}_{x,y,a,b}(3)$ is. To this aim, let us examine $\mathsf{eqdef}(\ell)$, namely :

$$\begin{aligned} 0 \equiv\ & X\big(x,y,-c+\ell)\,[e\,a+f\,b]+Y(x,y,-c+\ell)\,[g\,a+h\,b]-Z(x,y,-c+\ell) \\ & + A(a,b,c)\,[e\,x+g\,y]+B(a,b,c)\,[f\,x+h\,y]-C(a,b,c). \end{aligned}$$

Since the four multipliers $e\,a+f\,b$, $g\,a+h\,b$, $e\,x+g\,y$, $f\,x+h\,y$ are weighted *homogeneous* of order 1, the linear system obtained above splits into linear subsystems graded by the weighted order $\mu = 0, 1, 2, \dots$:

$$\big\{E^{\ell}_{i,j,l,m,n}(V_\mu) \,=\, 0\big\}_{i+j+l+m+2n=\mu},$$

which incorporate only the (homogeneous) variables V_μ. Solving $\mathsf{eqdef}(\ell)$ is equivalent to solving separately all these linear systems.

With a perturbation $G = \mathrm{O}_{x,y,a,b}(3)$:

$$\begin{aligned}
0 &\equiv \mathsf{eqdef}(\ell + G) \\
&\equiv X(x, y, -c + \ell + G)\,[\ell_x + G_x] \\
&\quad + Y(x, y, -c + \ell + G)\,[\ell_y + G_y] - Z(x, y, -c + \ell + G) \\
&\quad + A(a, b, c)\,[\ell_a + G_a] + B(a, b, c)\,[\ell_b + G_b] - C(a, b, c).
\end{aligned}$$

Then in view of :

$$2 \leqslant \mathsf{weighted\,order}\,\big(G_x,\, G_y,\, G_a,\, G_b\big),$$

we have for all $\mu \geqslant 0$, with *linear* dependence on written variables :

$$E^{\ell+G}_{i,j,l,m,n} = E^{\ell+G}_{i,j,l,m,n}\big(V_0, \dots, V_{\mu-1}, V_\mu\big) \qquad (i{+}j{+}l{+}m{+}2n = \mu),$$

and :

$$E^{\ell+G}_{i,j,l,m,n}\big(V_0, \dots, V_{\mu-1}, V_\mu\big)\Big|_{G=0} = E^{\ell+0}_{i,j,l,m,n}\big(V_0, \dots, V_{\mu-1}, V_\mu\big) = E^{\ell}_{i,j,l,m,n}\big(V_\mu\big).$$

Abbreviate :

$$E_\mu^\ell(V_\mu) \;:=\; \big\{E_{i,j,l,m,n}^\ell\big\}_{i+j+l+m+2n=\mu},$$
$$E_\mu^{\ell+G}\big(V_0,\dots,V_{\mu-1},V_\mu\big) \;:=\; \big\{E_{i,j,l,m,n}^{\ell+G}\big(V_0,\dots,V_{\mu-1},V_\mu\big)\big\}_{i+j+l+m+2n=\mu}.$$

From what precedes, it also follows :

Observation [Triangularity]. *For general $F = \ell + G$, the linear equations at order μ are a perturbation of the model linear equations :*

$$E_\mu^{\ell+G}\big(V_0,\dots,V_{\mu-1},V_\mu\big) \;=\; E_\mu^\ell(V_\mu) + R_\mu^{\ell+G}\big(V_0,\dots,V_{\mu-1}\big),$$

with a linear remainder vanishing identically $R_\mu^{\ell+0} \equiv 0$ when $G = 0$. □

So for any $\nu \geqslant 0$, the collection of all linear equations $\big\{E_\mu^{\ell+G} = 0\big\}_{0\leqslant\mu\leqslant\nu}$ up to order ν happens to be a triangular (by blocks) linear system :

$$\begin{aligned}
0 &= E_0^{\ell+G}\big(V_0\big) && = E_0^\ell\big(V_0\big),\\
0 &= E_1^{\ell+G}\big(V_0,V_1\big) && = E_1^\ell\big(V_1\big) + R_1^{\ell+G}\big(V_0\big),\\
0 &= E_2^{\ell+G}\big(V_0,V_1,V_2\big) && = E_2^\ell\big(V_2\big) + R_2^{\ell+G}\big(V_0,V_1\big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots && \cdots\cdots\cdots\cdots\cdots\cdots\\
0 &= E_\nu^{\ell+G}\big(V_0,V_1,V_2,\dots,V_{\nu-1},V_\nu\big) && = E_\nu^\ell\big(V_\nu\big) + R_\nu^{\ell+G}\big(V_0,V_1,V_2,\dots,V_{\nu-1}\big).
\end{aligned}$$

Our goal is to solve as many as possible variables V_0 from $0 = E_0^{\ell+G}(V_0)$, then as many as possible variables V_1 from $0 = E_1^{\ell+G}(V_0, V_1)$, then as many as possible variables V_2 from $0 = E_2^{\ell+G}(V_0, V_1, V_2)$, and so on. Clearly, by triangularity of the linear system, it suffices to understand how to do this only for the (truncated) homogeneous linear systems, without remainders at all, and with separated variables :

$$
\begin{aligned}
0 &= E_0^{\ell}\big(V_0\big), \\
0 &= E_1^{\ell}\big(V_1\big), \\
0 &= E_2^{\ell}\big(V_2\big), \\
&\cdots\cdots\cdots\cdots \\
0 &= E_\nu^{\ell}\big(V_\nu\big).
\end{aligned}
$$

Now, a count gives :

μ	$\#V_\mu$	$\#E_\mu$	Difference
0	2	1	1
1	8	4	4
2	16	11	5
3	28	24	4
4	42	46	1
5	60	80	0
6	80	130	0

Here, we understand :

$$\text{Difference} := \mathsf{max}\big(0,\ \#V_\mu - \#E_\mu\big).$$

The key explanation why $\mathsf{dim}\,\mathfrak{syn}(M) < \infty$ always is now made apparent by the fact that, starting from $\mu \geqslant 4$, there are (much) more equations than variables. On the other hand, for $\mu = 0, 1, 2, 3$, there are more variables than equations, hence not all variables can be solved.

In view of :

$$1 + 4 + 5 + 4 + 1 = 15,$$

one is conducted, after some experiments and reflection, to introduce 15 appropriate *parametric variables :*

$$V_{\mathsf{par}} := \left\{ \begin{matrix} & & & & & & C_{0,0,0}, \\ X_{0,0,0}, & Y_{0,0,0}, & A_{0,0,0}, & & B_{0,0,0}, & & \\ & & A_{1,0,0}, & A_{0,1,0} & B_{1,0,0}, & B_{0,1,0}, & C_{0,0,1} \\ X_{0,0,1}, & Y_{0,0,1}, & A_{0,0,1}, & & B_{0,0,1}, & & \\ & & & & & & C_{0,0,2} \end{matrix} \right\},$$

stratified by weights :

$$V_{\mu,\mathsf{par}} := V_{\mathsf{par}} \cap V_{\mu},$$

and to consider the other (remaining) variables as (potentially) *eliminable variables :*

$$V_{\mu,\mathsf{elm}} := V_{\mu} \Big\backslash V_{\mathsf{par}}.$$

Here is a count :

μ	$\#V_{\mu,\mathsf{par}}$	$\#V_{\mu,\mathsf{elm}}$	$\#V_\mu$
0	1	1	2
1	4	4	8
2	5	11	16
3	4	24	28
4	1	41	42
5	0	60	60
6	0	80	80

Here is a view of all variables in orders $\mu = 0, 1, 2, 3, 4$:

0			Z_{000}			C_{000}
1	X_{000}	Y_{000}	$Z_{100}Z_{010}$	A_{000}	B_{000}	$C_{100}C_{010}$
2	$X_{100}X_{010}$	$Y_{100}Y_{010}$	$Z_{200}Z_{110}Z_{020}$	$A_{100}A_{010}$	$B_{100}B_{010}$	$C_{200}C_{110}C_{020}$
			Z_{001}			C_{001}
3	$X_{200}X_{110}X_{020}$	$Y_{200}Y_{110}Y_{020}$	$Z_{300}Z_{210}Z_{120}Z_{030}$	$A_{200}A_{110}A_{020}$	$B_{200}B_{110}B_{020}$	$C_{300}C_{210}C_{120}C_{030}$
	X_{001}	Y_{001}	$Z_{101}Z_{011}$	A_{001}	B_{001}	$C_{101}C_{011}$
4	$X_{300}X_{210}X_{120}X_{030}$	$Y_{300}Y_{210}Y_{120}Y_{030}$	$Z_{400}Z_{310}Z_{220}Z_{130}Z_{040}$	$A_{300}A_{210}A_{120}A_{030}$	$B_{300}B_{210}B_{120}B_{030}$	$C_{400}C_{310}C_{220}C_{130}C_{040}$
	$X_{101}X_{011}$	$Y_{101}Y_{011}$	$Z_{201}Z_{111}Z_{021}$	$A_{101}A_{011}$	$B_{101}B_{011}$	$C_{201}C_{111}C_{021}$
			Z_{002}			C_{002}

In weight $\mu = 5$:

$X_{400}X_{310}X_{220}X_{130}X_{040}$	$Y_{400}Y_{310}Y_{220}Y_{130}Y_{040}$	$Z_{500}Z_{410}Z_{320}Z_{230}Z_{140}Z_{050}$	$A_{400}A_{310}A_{220}A_{130}A_{040}$	$B_{400}B_{310}B_{220}B_{130}B_{040}$	$C_{500}C_{410}C_{320}C_{230}C_{140}C_{050}$
$X_{201}X_{111}X_{021}$	$Y_{201}Y_{111}Y_{021}$	$Z_{301}Z_{211}Z_{121}Z_{031}$	$A_{201}A_{111}A_{021}$	$B_{201}B_{111}B_{021}$	$C_{301}C_{211}C_{121}Z_{031}$
X_{002}	Y_{002}	$Z_{102}Z_{012}$	A_{002}	B_{002}	$C_{102}Z_{012}$

In weight $\mu = 6$:

$$
\begin{array}{llllll}
X_{500}X_{410}X_{320}X_{230}X_{140}X_{050} & Y_{500}Y_{410}Y_{320}Y_{230}Y_{140}Y_{050} & Z_{600}Z_{510}Z_{420}Z_{330}Z_{240}Z_{150}Z_{060} & A_{500}A_{410}A_{320}A_{230}A_{140}A_{050} & B_{500}B_{410}B_{320}B_{230}B_{140}B_{050} & C_{600}C_{510}C_{420}C_{330}C_{240}C_{150}C_{060} \\
X_{301}X_{211}X_{121}X_{031} & Y_{301}Y_{211}Y_{121}Y_{031} & Z_{401}Z_{311}Y_{221}Z_{131}Z_{041} & A_{301}A_{211}A_{121}A_{031} & B_{301}B_{211}B_{121}B_{031} & C_{401}C_{311}C_{221}C_{131}C_{041} \\
X_{102}X_{012} & Y_{102}Y_{012} & Z_{202}Z_{112}Z_{022} & A_{102}A_{012} & B_{102}B_{012} & C_{202}C_{112}C_{022} \\
 & & Z_{003} & & & C_{003}
\end{array}
$$

However, this is not the full story. We now explain how to *select* appropriate linear equations, call them $E^{\ell}_{\mu,\mathsf{sel}}$, among the E^{ℓ}_{μ} for $\mu = 0, 1, 2, 3, 4, 5, 6, \dots,$ in order to solve all these $V_{\mu,\mathsf{elm}}$ in terms of the parametric variables $V_{\mu,\mathsf{par}}$ only, so that :

$$
\#E^{\ell}_{\mu,\mathsf{sel}} = \#V_{\mu,\mathsf{sel}} \qquad\qquad (\forall\, \mu \geqslant 0).
$$

At first, in low orders $\mu = 0, 1, 2, 3$, we take all equations :

$$
\begin{aligned}
E^{\ell}_{0,\mathsf{sel}} &:= E^{\ell}_{0}, \\
E^{\ell}_{1,\mathsf{sel}} &:= E^{\ell}_{1}, \\
E^{\ell}_{2,\mathsf{sel}} &:= E^{\ell}_{2}, \\
E^{\ell}_{3,\mathsf{sel}} &:= E^{\ell}_{3}.
\end{aligned}
$$

Lemme. *For* $\mu = 0, 1, 2, 3$*, for any rank* 2 *Levi matrix* $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$*, the linear system :*

$$0 = E^{\ell}_{\mu,\mathsf{sel}}\Big(V_{\mu,\mathsf{par}},\ V_{\mu,\mathsf{elm}}\Big) \qquad\qquad (\mu = 0,1,2,3),$$

is Cramer with respect to $V_{\mu,\mathsf{elm}}$*.*

Definition. A linear system of $\mathrm{Q} \geqslant 1$ equations in $\mathrm{P} + \mathrm{Q}$ variables $\big(x_1, \dots, x_{\mathrm{P}},\ y_1, \dots, y_{\mathrm{Q}}\big)$ with $\mathrm{P} \geqslant 0$:

$$0 = \Lambda_\tau\big(x_1, \dots, x_{\mathrm{P}},\ y_1, \dots, y_{\mathrm{Q}}\big) \qquad\qquad (1 \leqslant \tau \leqslant \mathrm{Q}),$$

is called *Cramer with respect to* $(y_1, \dots, y_{\mathrm{Q}})$ if it has nonzero determinant with respect to the variables $(y_1, \dots, y_{\mathrm{Q}})$:

$$0 \neq \Big(\frac{\partial \Lambda_\tau}{\partial y_\chi}\Big)^{1 \leqslant \tau \leqslant \mathrm{Q}}_{1 \leqslant \chi \leqslant \mathrm{Q}}.$$

When the coefficients of the linear forms Λ_τ depend on parameters, *e.g.* the coefficients e, f, g, h of the Levi form at the origin, and/or some parameters α, β, γ, …upon which $F = F_{\alpha,\beta,\gamma,\dots}(x, y, a, b)$ may depend, we require that the determinant in question is *nonzero for any value of all the parameters.*

Next, in weights $\mu \geqslant 4$, we really have to select appropriate equations, because there are too many equations. The last step, to be discussed later, will be to examine what additional linear relations the unselected equations bring as major supplementary constraints on the coefficients of $L = X\,\partial_x + Y\,\partial_y + Z\,\partial_z + A\,\partial_a + B\,\partial_b + C\,\partial_c$ to really be an infinitesimal symmetry of $\big\{z = -c + F\big\}$.

For instance, when $\mu = 4$, we have :

$$42 = \#V_\mu \qquad\qquad \text{while} \qquad\qquad \#E_\mu = 46.$$

However, an examination of this system $0 = E_4^\ell(V_4)$ of 46 equations in 42 variables shows that its rank is not equal to 42, but to 41. Therefore, only 41 variables can be eliminated. In weights $\mu \geqslant 5$, the ranks of the system $0 = E_\mu^\ell(V_\mu)$ is always equal to the number $\#V_\mu$ of variables.

μ	$\#E_\mu$	$\#V_\mu$	Rank
4	46	42	41
5	80	60	60
6	130	80	80

We now want to select precisely 41, 60, 80 equations among E_4^ℓ, E_5^ℓ, E_6^ℓ.

To begin with, we perform a preliminary sub-selection, which works for all rank 2 Levi matrices $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$, *i.e.* with $0 \neq ef - gh$. However, the complete selection $E_{\mu,\mathsf{sel}}^\ell \subset E_\mu^\ell$ will require a discussion, depending on whether some of the coefficients e, f, g, h vanish.

Let us introduce 6 branches of selected eqations :

$$
\begin{aligned}
B1_\mu^\ell &:= \big\{E_{i,j,0,0,n}^\ell\big\}_{i+j+2n=\mu}, \\
B2_\mu^\ell &:= \big\{E_{i,j,1,0,n}^\ell\big\}_{i+j+2n=\mu-1}, \\
B3_\mu^\ell &:= \big\{E_{i,j,0,1,n}^\ell\big\}_{i+j+2n=\mu-1}, \\
B4_\mu^\ell &:= \big\{E_{0,0,l,m,n}\big\}_{l+m+2n=\mu}, \\
B5_\mu^\ell &:= \big\{E_{1,0,l,m,n}^\ell\big\}_{l+m+2n=\mu-1} \\
B6_\mu^\ell &:= \big\{E_{0,1,l,m,n}^\ell\big\}_{l+m+2n=\mu-1},
\end{aligned}
$$

and define :

$$B_\mu^\ell := \bigcup_{1\leqslant I\leqslant 6} BI_\mu^\ell.$$

For $\mu = 0, 1, 2, 3$, one can verify that :

$$B_\mu^\ell = E_\mu^\ell \qquad\qquad (\mu = 0,1,2,3).$$

A count gives :

μ	$\#BI_\mu^\ell$	$\#V_\mu$	Difference
4	37	42	5
5	56	60	4
6	75	80	5

Proposition. *Whatever rank* 2 *Levi matrix* $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$*, for every weight* $\mu \geqslant 4$*, the linear sub-system :*

$$0 = BI_\mu^\ell(V_\mu),$$

is Cramer with respect to the variables :

$$V_{\mu,\mathsf{elm}}^B := V_{\mu,\mathsf{elm}} \Big\backslash V_{\mu,\mathsf{spo}}. \qquad \square$$

Here are the remaining *sporadic variables* $V_{\mu,\mathsf{spo}}$, shown weight by weight :

μ							$\#$
4		$A_{1,0,1}$,	$A_{0,1,1}$,	$B_{1,0,1}$,	$B_{0,1,1}$,		4
5	$X_{0,0,2}$,	$Y_{0,0,2}$,		$A_{0,0,2}$,		$B_{0,0,2}$,	4
6		$A_{1,0,2}$,	$A_{0,1,2}$,	$B_{1,0,2}$,	$B_{0,1,2}$,	$C_{0,0,3}$,	5
7	$X_{0,0,3}$,	$Y_{0,0,3}$,		$A_{0,0,3}$,	$B_{0,0,3}$,		4
8		$A_{1,0,3}$,	$A_{0,1,3}$,	$B_{1,0,3}$,	$B_{0,1,3}$,	$C_{0,0,4}$,	5

Next, we have to select some sporadic linear equations among all the E_μ^ℓ, in number 4, 4, 5, for $\mu = 4, 5, 6$, and so on [after weight 6, everything is easy]. However, unfortunately, the selection will depend on $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. We single out a few useful cases.

- Levi matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ with $e, f, g, h \neq 0$:

$$E_{4,\mathsf{spo}}^\ell := \big\{E_{2,0,2,0,0},\ E_{2,0,0,2,0},\ E_{0,2,2,0,0},\ E_{0,2,0,2,0}\big\},$$
$$E_{5,\mathsf{spo}}^\ell := \big\{E_{3,0,2,0,0},\ E_{0,2,3,0,0},\ E_{0,3,2,0,0},\ E_{0,2,0,3,0}\big\},$$
$$E_{6,\mathsf{spo}}^\ell := \big\{E_{3,0,3,0,0},\ E_{3,0,0,3,0},\ E_{0,3,3,0,0},\ E_{0,3,0,3,0},\ E_{2,0,2,0,1}\big\}.$$

- Levi matrix $\left(\begin{smallmatrix} e & 0 \\ 0 & h \end{smallmatrix}\right)$ with $e, h \neq 0$:

$$
\begin{aligned}
E^{\ell}_{4,\mathsf{spo}} &:= \left\{E_{2,0,2,0,0},\ E_{2,0,1,1,0},\ E_{0,2,0,2,0},\ E_{0,2,1,0,1}\right\}, \\
E^{\ell}_{5,\mathsf{spo}} &:= \left\{E_{3,0,2,0,0},\ E_{0,3,0,2,0},\ E_{0,2,0,3,0},\ E_{2,0,3,0,0}\right\}, \\
E^{\ell}_{6,\mathsf{spo}} &:= \left\{E_{3,0,3,0,0},\ E_{3,0,2,1,0},\ E_{0,3,1,2,0},\ E_{0,3,0,3,0},\ E_{2,0,2,0,1}\right\}.
\end{aligned}
$$

- Levi matrix $\begin{pmatrix} 0 & f \\ g & h \end{pmatrix}$ with $f, g, h \neq 0$:

$$
\begin{aligned}
E^{\ell}_{4,\mathsf{spo}} &:= \left\{E_{2,0,1,1,0},\ E_{2,0,0,2,0},\ E_{0,2,2,0,0},\ E_{1,1,0,2,0}\right\}, \\
E^{\ell}_{5,\mathsf{spo}} &:= \left\{E_{3,0,0,2,0},\ E_{0,2,3,0,0},\ E_{0,3,2,0,0},\ E_{0,2,0,3,0}\right\}, \\
E^{\ell}_{6,\mathsf{spo}} &:= \left\{E_{3,0,0,3,0},\ E_{0,3,3,0,0},\ E_{0,3,0,3,0},\ E_{3,0,1,2,0},\ E_{0,2,0,2,1}\right\}.
\end{aligned}
$$

Finally, we set :

$$
E^{\ell}_{\mu,\mathsf{sel}} := B^{\ell}_{\mu} \cup E^{\ell}_{\mu,\mathsf{spo}}.
$$

Proposition. *For every weight $\mu \geqslant 4$, the system $E^{\ell}_{\mu,\mathsf{sel}}(V_\mu) = 0$ is Cramer with respect to $V_{\mu,\mathsf{elm}}$.*

Consequently, one can solve :

$$
V_{\mu,\mathsf{elm}} = \mathsf{Linear}\left(V_{\mu,\mathsf{par}}\right) \qquad (\mu \geqslant 4).
$$

For L to really belong to $\mathfrak{sym}\big(\{z=-c+\ell\}\big)$, all equations $E_\mu^\ell(V_\mu)=0$ should be satisfied. Now, a count gives :

μ	$\#E_\mu$	$\#E_{\mu,\mathsf{sel}}$	Difference
4	46	41	5
5	80	60	20
6	130	80	50

Proposition. *For the model $\{z=-c+\ell\}$, all other equations $E_\mu \backslash E_{\mu,\mathsf{sel}}$ are automatically satisfied.* □

Question. *What about an arbitrary submanifold $\{z=-c+\ell+G\}$?*

Thanks to triangularity (by blocks) of the linear systems, precisely the same linear equations can be selected for arbitrary remainder $G=\mathrm{O}_{x,y,a,b}(3)$. Hence we define/select, for $\mu\geqslant 0$:

$$B_\mu^{\ell+G} := \bigcup_{1\leqslant I\leqslant 6} BI_\mu^{\ell+G},$$

$$E_{\mu,\mathsf{sel}}^{\ell+G} := B_\mu^{\ell+G}\cup E_{\mu,\mathsf{sel}}^{\ell+G}.$$

By convention, for low order $\mu = 0, 1, 2, 3$:

$$\emptyset \;=:\; E^{\ell+G}_{\mu,\mathsf{spo}} \;=\; E^{\ell}_{\mu,\mathsf{spo}}.$$

The final selected triangular systems are, for any order $\nu \geqslant 0$:

$$E^{\nu,\mathsf{sel}}_{\ell+G} \;:=\; \bigcup_{0\leqslant\mu\leqslant\nu} E^{\ell+G}_{\mu,\mathsf{sel}}.$$

The variables are :

$$\begin{aligned} V^{\nu} \;&:=\; \bigcup_{0\leqslant\mu\leqslant\nu} V_{\mu}, \\ V^{\nu,\mathsf{elm}} \;&:=\; \bigcup_{0\leqslant\mu\leqslant\nu} V_{\mu,\mathsf{elm}}, \\ V^{\nu,\mathsf{par}} \;&:=\; V^{\mathsf{par}} \cap V^{\nu}. \end{aligned}$$

A counts shows :

μ	$\#V^{\nu,\mathsf{elm}}$	$\#E^{\nu,\mathsf{sel}}$	Difference
0	1	1	0
1	5	5	0
2	16	16	0
3	40	40	0
4	81	81	0
5	141	141	0
6	221	221	0

Proposition [Cramer complete system]. *For every $\nu \geqslant 0$, the system :*

$$0 = E^{\nu,\mathsf{sel}}\big(V^\nu\big),$$

can be solved uniquely for the variables $V^{\nu,\mathsf{elm}}$, in terms of $V^{\nu,\mathsf{par}}$. □

In particular, for $\nu \geqslant 4$, since $V^{\nu,\mathsf{par}} = V^{\mathsf{par}}$, on Maple, we get formulas :

$$V^{\nu,\mathsf{elm}} = \mathsf{Linear}\,\big(V^\nu\big) \qquad (\nu \geqslant 4).$$

Finally, substituting back these solutions in all (remaining) equations when $\nu \geqslant 4$:

$$\bigcup_{4 \leqslant \mu \leqslant \nu} E_{\mu}^{\ell+G} \Big\backslash E_{\mu,\mathsf{sel}}^{\ell+G},$$

we get a number of significant equations. A count gives :

ν	$\#E^{\nu} \backslash E^{\nu,\mathsf{sel}}$
4	5
5	25
6	75

We may analyze these significant equations on Maple. All are of the form :

$$\begin{aligned} 0 = \; & \Lambda_1 \, C_{0,0,0} \\ & \Lambda_2 \, X_{0,0,0} + \Lambda_3 \, Y_{0,0,0} + \Lambda_4 \, A_{0,0,0} + \Lambda_5 \, B_{0,0,0} \\ & + \Lambda_6 \, A_{1,0,0} + \Lambda_7 \, A_{0,1,0} + \Lambda_8 \, B_{1,0,0} + \Lambda_9 \, B_{0,1,0} + \Lambda_{10} \, C_{0,0,1} \\ & + \Lambda_{11} \, X_{0,0,1} + \Lambda_{12} \, Y_{0,0,1} + \Lambda_{13} \, A_{0,0,1} + \Lambda_{14} \, B_{0,0,1} \\ & + \Lambda_{15} \, C_{0,0,2}, \end{aligned}$$

where $\Lambda_1, \dots, \Lambda_{15}$ depend ultimately on $F = \ell + G$, in some complicated algebraic way. So if $F = F_{\alpha,\beta,\gamma,\dots}$ depends on parameters, we get a systems of 5, 25, 75 linear equations in 15 variables.

To solve these equations for the tubifications of the homogeneous DKR surfaces, we employ the Gauss-pivot method, manually on Maple.