Moving frames and invariants for submanifolds in parabolic homogeneous spaces

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Outline

1. **The problem and initial examples**
   - Definitions
   - Lifting the submanifolds to correspondence spaces
   - Example: symbols of curves in G_2 geometry

2. **Prototype: projective geometry of plane curves**
   - 19-th century
   - Premier example of moving frame construction
   - Reduction procedure
   - Parametric computation in Maple

Integral curves in parabolic homogeneous spaces

- Let $M = G/P$ be an arbitrary parabolic homogeneous space: $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a graded semisimple Lie algebra of the Lie group G and $\mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$.

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- Given a submanifold $N \subset M$ we define its symbol at $x \in N$ as $\operatorname{gr} T_x N$ viewed as a graded subspace (actually a subalgebra) in $\mathfrak{g}_-$.
- To be more precise, choose $g \in G$ such that $g.x = o$ and consider $g_*(T_x N)$ as a subspace of $T_o M$, which can be naturally identified with $\mathfrak{g}_-$. As g is defined only modulo left multiplication by P, the subspace $g_*(T_x N)$ is also defined modulo the adjoint action of P on $\mathfrak{g}_- \equiv \mathfrak{g}/\mathfrak{p}$. Then $\operatorname{gr} g_*(T_x N)$ is well-defined modulo the action of G_0 on $\mathfrak{g}_-$.

Assumption of constant symbol

- Let $\mathfrak{n}$ be a graded subalgebra in $\mathfrak{g}_-$. We say that N has constant symbol $\mathfrak{n}$, if $\operatorname{gr} T_x N$ is G_0-equivalent to $\mathfrak{n}$ for any $x \in N$. In the following we shall always assume that N has a constant symbol, *which is the only assumption on* N.

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- (Intrinsic) prolongation of $\mathfrak{n}$ in $\mathfrak{g}$ is a largest graded subalgebra $\operatorname{Prol}(\mathfrak{n})$ of $\mathfrak{g}$ such that $\operatorname{Prol}_-(\mathfrak{n}) = \mathfrak{n}$. It can be constructed inductively as:

$$\operatorname{Prol}_i(\mathfrak{n}) = \mathfrak{n}_i, \quad (i < 0),$$
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- **Theorem**. *We have* $\dim \operatorname{sym}(N) \leq \dim \operatorname{Prol}(\mathfrak{n})$. *Moreover the equality is achieved if and only if N is locally equivalent to the orbit of the subgroup* $\exp \operatorname{Prol}(\mathfrak{n}) \subset G$ *through $o = eP$.*

Hypersurfaces in projective spaces

- Assume that G/P is the projective space P^n, and $\dim N = n - 1$. Here $G = PGL(n+1, \mathbb{R})$ and $P = P_1$. We identify $\mathfrak{g}_- = \mathfrak{g}_{-1}$ with $\mathbb{R}^n$ as follows:

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- The most symmetric submanifold with this symbol is just a hyperplane in P^n. Note that its symmetry group is yet another parabolic subgroup P_n in G. Note that the intersection $P_{1,n} = P_1 \cap P_n$ is another parabolic. Note that $G/P_{1,n}$ can be naturally identified with the jet space $J^1(P^n, n-1)$ of 1-jet of hypermanifolds in P^n.

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- Looking now at possible symbols of Lagrangian submanifolds in $G/P_{1,n}$, we see that any such symbol has the form:

$$\mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & (SY)^t & 0 \end{pmatrix} \quad Y \in \mathrm{Mat}_{n-1,1}(\mathbb{R}),$$

  where S is an arbitrary symmetric matrix. So, classifying the symbols is equivalent to classifying the symmetric $(n-1)x(n-1)$ matrices up to the action of the (new) $G_0 = GL(n-1, \mathbb{R})$. It is easy to see that geometrically S corresponds to the second fundamental form of N.

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- In particular, if $n = 2$ we can assume that:

$$\mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ 0 & y & 0 \end{pmatrix}; \quad \mathrm{Prol}(\mathfrak{n}) = \begin{pmatrix} h & x & 0 \\ y & 0 & x \\ 0 & y & -h \end{pmatrix} \quad x, y, h \in \mathbb{R},$$

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  which the symmetry algebra of a non-degenerate conic in P^2.
- Such lift of N to a larger parabolic geometry (correspondence space) works whenever the prolongation of the initial symbol is a paraboic subalgebra.

Example: symbols of curves in G_2 geometry

Classical approach of Sophus Lie

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$$y_2 = 0, \quad \text{(equation on stright lines)}$$
$$9y_2^2 y_5 - 45 y_2 y_3 y_4 + 40 y_3^2 = 0, \quad \text{(equation on all conics)}.$$

Here we use the standard coordinate system $(x, y, y_1, \dots, y_k)$ on the jet space $J^k(\mathbb{R}, \mathbb{R})$.

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- Encode a plane curve via a 3rd order linear differential equation:

$$u''' + p_2(t)u'' + p_1(t)u' + p_0(t)u = 0$$

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- Use change of variables $(t, u) \mapsto (\lambda(t), \mu(t)u)$ to kill the coefficients p_2 and p_1. This is always possible, but requires in general solving Riccati equation. Then the remaining coefficient $p_0(t)$ would define a 5-th order relative invariant of the curve.

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- Explicitly, for a curve parametrized in affine coordinates as $(t, u(t))$ one needs to define a 3rd order linear ODE whose solution space is $\{1, t, u(t)\}$ and then go through the normalization of p_1 and p_2. The recovered expression for p_0 coincides with the above 5th order relative invariant computed by Sophus Lie.

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- For $N \subset M$ define $Q_{-1} = \pi^{-1}(N) \subset G$. Then $\pi \colon Q_{-1} \to N$ is a principal P-bundle over N, which conststs of ell *frames* over N. A moving frame for $N \subset M$ is just an arbitrary section $s \colon N \to Q_{-1}$.

Normalizing frames

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$$s^*\omega = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} + \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix}.$$

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- Here σ is a certain 1-form on the so far unparametrized curve. The *arc-length* parametrization of the curve is just a choice of a local coordinate s such that $\sigma = ds$.

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- The second summand defines the relative projective invariant $k = \omega_{02}/\omega_{10}$. The plane curve is locally a conic if and only if $k = 0$.

Frame reductions

- The construction of the above frame bundle is performed iteratively. As above, we first lift N to $N^{(1)} \subset J^1(P^2, 1) = G/P_{1,2}$ and define Q_{-1} as $\pi^{-1}(N^{(1)})$. Then we know that $\omega_{-1} = \omega|_{Q_{-1}}$ takes values in

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- it is easy to see that ω_{21} vanishes identically iff our curve is a straight line. Assume that $\omega_{21} \neq 0$ and define Q_0 as all points $z \in Q_{-1}$ where ω_{-1} takes values in

$$\begin{pmatrix} * & * & * \\ \omega_{10} & * & * \\ 0 & \omega_{10} & * \end{pmatrix}.$$

  The set of all such points z forms a reduced principal bundle with the structure group

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} \\ 0 & 1 & p_{12} \\ 0 & 0 & p_{00}^{-1} \end{pmatrix}.$$

Further reductions

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- Finally, Q_2 is defined as all $z \in Q_1$, where $\omega_1 = \omega|_{Q_1}$ takes values in

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