Moving frames and invariants for submanifolds in parabolic homogeneous spaces
Lecture 3

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GRIEG seminar, December 3, 2021

Outline

1. **Revised definitions**
   - Revised definitions
   - Constant symbol assumption
   - Lifting the submanifolds to correspondence spaces

2. **Construction of the canonical moving frame**
   - Normalization conditions
   - Existence of the normal moving frame

3. **Applications**
   - Deformations of rational homogeneous varieties
   - Infinite-dimensional case

Submanifolds in parabolic homogeneous spaces

- Let $M = G/P$ be an arbitrary parabolic homogeneous space: $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a graded semisimple Lie algebra of the Lie group G and $\mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$.
- M is naturally equipped with a a structure of a filtered manifold

$$0 \subset T^{-1}M \subset \cdots \subset T^{-\nu}M = TM$$

defined as a flag of G-invariant vector distributions equal to $\oplus_{i\leq k}\mathfrak{g}_{-i}$ mod $\mathfrak{p}$ at $o = eP$.

- Given a submanifold $N \subset M$ we define its symbol at $x \in N$ as $\operatorname{gr} T_x N$ viewed as a graded subspace in $\mathfrak{g}_-$.
- The symbol is a graded subalgebra in $\mathfrak{g}_-$, viewed up to the action of G_0. In general, it depends on a point $x \in N$.

Assumption of constant symbol

- Let $\mathfrak{n}$ be a graded subalgebra in $\mathfrak{g}_-$. We say that N has constant symbol $\mathfrak{n}$, if $\operatorname{gr} T_x N$ is G_0-equivalent to $\mathfrak{n}$ for any $x \in N$. In the following we shall always assume that N has a constant symbol, *which is the only assumption on N*.
- Unlike the case of curves, the assumption of constant symbol for submanifolds of dimension ≥ 2 is restrictive.
- **Example 1.** Generic r-dimensional submanifolds in $\mathrm{Gr}(k, n)$ will not have a constant symbol. Indeed, the symbol in this case is an r-dimensional subspace in $\mathrm{Mat}(k, n-k)$, viewed up to the action of $SL(k) \times SL(n-k) \subset G_0$. For dimensional reasons (eg. already for 3-dim submanifolds in $\mathrm{Gr}(3, 6)$) such orbits have continuous parameters, that become functional invariants for a generic submanifold.

Submanifolds in projective spaces

- Assume that G/P is the projective space P^n, and $\dim N = r$. Here $G = PGL(n+1, \mathbb{R})$ and $P = P_1$. We identify $\mathfrak{g}_- = \mathfrak{g}_{-1}$ with $\mathbb{R}^n$ as follows:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \subset \mathfrak{gl}(n+1, \mathbb{R}), \quad X \in \mathrm{Mat}_{n,1}(\mathbb{R}).$$

- It is clear that $G_0 = GL(n, \mathbb{R})$ acts transitively on all subspaces of dimension r in $\mathfrak{g}_{-1}$. Let us fix $\mathfrak{n}$ as a subspace spanned by the first r coordinate vectors in $\mathbb{R}^n$:

$$\mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subset \mathfrak{g}_{-1}, \quad Y \in \mathrm{Mat}_{r,1}(\mathbb{R}).$$

- The most symmetric submanifold with this symbol is just an r-dim linear space in P^n. Note that its symmetry group is yet another parabolic subgroup P_n in G. The intersection $P_{1,r+1} = P_1 \cap P_{r+1}$ is another parabolic. The space $G/P_{1,r+1}$ can be naturally identified with the jet space $J^1(P^n, r)$ of 1-jets of r-dim submanifolds in P^n.

- We have a natural lift of $N \hookrightarrow G/P_{1,r+1}$, which takes N into its 1-st jet $j^1 N$. It is clear that (local) projective geometry of submanifolds in P^n coincides with the (local) geometry of r-dim submanifolds in $M = G/P_{1,r+1} = J^1(P^n, r)$, which are tangent to $T^{-1}M$ and are transversal to the fibers of the projection $J^1(P^n, r) \to P^n$.
- Looking now at possible symbols of such submanifolds in $G/P_{1,r+1}$, we see that any such symbol has the form:

$$\mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & (S_1 Y)^t & 0 \\ 0 & \vdots & 0 \\ 0 & (S_{n-r} Y)^t & 0 \end{pmatrix} \qquad Y \in \mathrm{Mat}_{r,1}(\mathbb{R}),$$

  where $S_1, \dots, S_{n-r}$ are arbitrary symmetric matrices.
- Classifying the symbols is equivalent to classifying subspaces $\langle S_1, \dots, S_{n-r} \rangle$ in the space of symmetric r by r matrices up to the action of $GL(r, \mathbb{R})$.
- Already in the case of 3-dim submanifolds in P^6 we get continuous parameters.

- **Example 2.** Consider symbols of submanifolds in contact G_2 geometry $M = G_2/P_2$. Two-dimensional contact (Legendrian) submanifolds have a symbol, which is a 2-dim Legendrian subspace in the 4-dim symplectic space $\mathfrak{g}_{-1}$ viewed up to the action of $G_0 = GL(2)$. There is only a finite number of orbits of this action, so we can assume that sub submanifold has a constant symbol.
- However, if we consider 3-dim submanifolds, then their symbol is a 3-dim graded subalgebra $\mathfrak{n} = \mathfrak{n}_{-2} + \mathfrak{n}_{-1}$ in $\mathfrak{g}_{-}$, where $\mathfrak{n}_{-2} = \mathfrak{g}_{-2}$ and $\mathfrak{n}_{-1}$ is *an arbitrary* 2-dim subspace in $\mathfrak{g}_{-1}$. The action of $GL(2)$ on $\mathrm{Gr}(2, 4)$ has continuous invariants.

Intrinsic prolongation

- (Intrinsic) prolongation of $\mathfrak{n}$ in $\mathfrak{g}$ is a largest graded subalgebra $\mathrm{Prol}(\mathfrak{n})$ of $\mathfrak{g}$ such that $\mathrm{Prol}_-(\mathfrak{n}) = \mathfrak{n}$. It can be constructed inductively as:

$$\mathrm{Prol}_i(\mathfrak{n}) = \mathfrak{n}_i, \quad (i < 0),$$
$$\mathrm{Prol}_i(\mathfrak{n}) = \{u \in \mathfrak{g}_i \mid [\mathfrak{n}, u] \subset \oplus_{j<i} \mathrm{Prol}_j(\mathfrak{n})\}, \quad (i \geq 0).$$

- **Theorem**. *We have* $\dim \mathrm{sym}(N) \leq \dim \mathrm{Prol}(\mathfrak{n})$. *Moreover the equality is achieved if and only if* N *is locally equivalent to the orbit of the subgroup* $\exp \mathrm{Prol}(\mathfrak{n}) \subset G$ *through* $o = eP$.
- Notation:

$$\mathrm{Prol}(\mathfrak{n})^{(0)} = \sum_{i \geq 0} \mathrm{Prol}_i(\mathfrak{n}).$$

This is a subalgebra of $\mathrm{Prol}(\mathfrak{n})$ that corresponds to the stationary subalgebra of the flat model.

Notion of a normal moving frame

- $N \subset G/P$ is any submanifold with a constant symbol $\mathfrak{n} \in \mathfrak{g}_-$
- $\pi \colon G \to G/P$ is the standard principle P-bundle, and $Q_{-1} = \pi^{-1}(N)$
- $\omega \colon TG \to \mathfrak{g}$ is the left-invariant Maurer–Cartan form on G
- $\pi|_{Q_{-1}} \colon Q_{-1} \to \gamma$ is the restriction of the principle P-bundle to γ
- *Moving frame* is (any) subbundle of this bundle: $E \subset Q_{-1}$
- We construct a *normal moving frame* $\pi \colon Q \to N$ by imposing conditions on $\omega(T_z Q) \subset \mathfrak{g}$ for $z \in Q$

Normalization conditions

- Define normalization conditions for any symbol $\mathfrak{n} \subset \mathfrak{g}_-$ as $\operatorname{Prol}(\mathfrak{n})^{(0)}$-invariant subspaces $W_1 \subset \mathfrak{g}$ and $W_2 \subset C^1_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n}))$ such that:

$$\mathfrak{g} = \operatorname{Prol}(\mathfrak{n}) \oplus W_1;$$
$$C^1_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) = \partial C^0_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) \oplus W_2.$$

- If $\operatorname{Prol}(\mathfrak{n})$ is reductive, then we can take W_1 as the complement to $\operatorname{Prol}(\mathfrak{n})$ with respect to the Killing form of $\mathfrak{g}$.
- Next, one can prove that there exists a scalar product $(\,,\,)$ on $\mathfrak{g}$ preserved by the adjoint action of $\operatorname{Prol}(\mathfrak{n})^{(0)}$. It can be used to define the codifferential

$$\partial^* \colon C^i(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) \to C^{i-1}(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n}))$$

dual to the standard Lie algebra cohomology operator ∂. Then one can define $W_2 = \ker \partial^*$.

Main result

Theorem

Fix normalization conditions W_1, W_2 *for a symbol* $\mathfrak{n} \in \mathfrak{g}_-$. *Then there exists a unique moving frame* $Q \to N$ *satisfying the following normalization condition. Decompose* $\omega|_Q$ *as* $\omega_I + \omega_{II}$ *according to the decomposition* $\mathfrak{g} = \operatorname{Prol}(\mathfrak{n}) \oplus W_1$. *Then:*

1. ω_I *defines a Cartan connection on* N *modelled by* $\operatorname{Prol}(\mathfrak{n}) / \operatorname{Prol}(\mathfrak{n})^{(0)}$. *It defines induced intrinsic geometry on* N.
2. $\omega_{II} = \chi \circ \omega_I$, *where* $\chi \in W_2$. *Here* χ *is viewed as an element of* $\operatorname{Hom}(\mathfrak{n}, W_1) = C^1(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n}))$.
3. *the part of* χ *taking values in* $H^1_+(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})) \cong W_2 \cap \ker \partial$ *defines fundamental invariants of the embedding.*

Explaining normalization conditions

- As mentioned before, we would like to normalize the moving frame by imposing *linear* conditions on the image of the Maurer–Cartan form ω.
- The condition of ω_I to be a Cartan connection implies that $\operatorname{Im}\omega \supset \operatorname{Im}\omega_I = \operatorname{Prol}(\mathfrak{n})$.
- The condition $\omega_{II} = \chi \circ \omega_I$ essentially means that

$$\operatorname{Im}\omega = \{X + \chi(X) \mid X \in \mathfrak{n}\} + \operatorname{Prol}(\mathfrak{n})^{(0)}.$$

- So, imposing the (linear) normalization conditions on χ, such as, for example, $\partial^*\chi = 0$ is equivalent to imposing (linear) normalization conditions on $\operatorname{Im}\omega$.

- T. Morimoto, Yu. Machida, B.D. *Extrinsic geometry and linear differential equations*, SIGMA, 17, Paper 061, (2021). `arXiv:1904.05687`.
- The idea is to start from the principal P-bundle $\pi\colon Q_{-1} \to N$, where $Q_{-1} = \pi^{-1}(N)$ and reduce it to a series of principal subbundles $\pi_k\colon Q_k \to N$, each with its own structure group having the Lie algebra

$$\sum_{i=0}^{k} \operatorname{Prol}_i(\mathfrak{n}) + \sum_{j>k} \mathfrak{g}_j.$$

- At each step $k \geq 0$ we define Q_k as the set of all such points $z \in Q_{k-1}$, that

$$(\omega_z)_{II} = \chi \circ (\omega_z)_I, \quad \text{where } \chi \in W_2 + \sum_{i>k} C^1_i(\mathfrak{n}, \mathfrak{g}/\operatorname{Prol}(\mathfrak{n})).$$

Algebra and moving frames of submanifolds

Algebra	**Geometry**
Graded subalgebra $\mathfrak{n} \subset \mathfrak{g}_-$	Submanifold N of a parabolic homogeneous space with constant symbol
Intrinsic prolongation $\mathrm{Prol}(\mathfrak{n})$	Symmetry algebra of the flat model
Normalization conditions: $\mathfrak{g} = \mathrm{Prol}(\mathfrak{n}) \oplus W_1$, $C^1_+(\mathfrak{n}, \mathfrak{g}/\mathrm{Prol}(\mathfrak{n})) = \partial C^0_+ \oplus W_2$	Canonical moving frame
$H^1_+(\mathfrak{n}, \mathfrak{g}/\mathrm{Prol}(\mathfrak{n}))$	Fundamental invariants of the submanifold

Rational homogeneous varieties

- Let S be a complex semisimple Lie group and let V be its irreducible representation. Let $\mathfrak{s}$ be the Lie algebra of S. Fix a parabolic $S^0 \subset S$. Let $\mathfrak{s} = \sum_{i=-\mu}^{\mu}$ be the corresponding grading of $\mathfrak{s}$, and let $e \in \mathfrak{s}_0$ be the grading element. Its action induces the compatible grading on V.
- The action of S on the standard flag of V induces the embedding $S/S^0 \to \mathrm{Flag}_\alpha(V)$.
- **Example:** the unique closed orbit (along with its osculating flag) of S acting irreducibly on $P(V)$. Such embeddings are know as *rational homogeneous varieties*.
- It is easy to see that the submanifold $S/S^0 \subset \mathrm{Flag}_\alpha(V)$ is flat.
- One can prove that the intrinsic prolongation $\mathrm{Prol}(\mathfrak{s}_-)$ in $\mathfrak{sl}(V)$ coincides with $\mathfrak{s}$.

- Computation of $H^1_+(\mathfrak{s}_-, \mathfrak{sl}(V)/\mathfrak{s})$ can be done via Kostant theorem. A rational homogeneous variety is said to be *rigid*, if this cohomology is trivial.
- Equivalently: any submanifold in $\mathrm{Flag}_\alpha(V)$ with the same symbol as $S/S^0 \hookrightarrow \mathrm{Flag}_\alpha(V)$ is locally equivalent to S/S^0.

Theorem

The only possible non-rigid rational homogeneous varieties are P^ℓ, Q^ℓ, $F_{1,\ell}(\mathbb{C}^{\ell+1})$ or, if S is not simple, those having these varieties in the direct product decomposition. (cf. Hwang–Yamaguchi, Landsberg–Robles)

Ideas of the proof

- Let $\mathfrak{s} = \oplus \mathfrak{s}_i$ be a complex simple graded Lie algebra and U an irreducible submodule of the $\mathfrak{s}$-module $\mathfrak{sl}(V)$. The cohomology group $H^1_r(\mathfrak{s}_-, U)$ vanishes for $r \geq 1$ except for the following cases:
  1. (A_3, Σ) with $\Sigma = \{\alpha_2\}$,
     (A_l, Σ) $(l \geq 1)$ with $\Sigma = \{\alpha_1\}$, $\{\alpha_l\}$, or $\{\alpha_1, \alpha_l\}$,
  2. (B_2, Σ) with $\Sigma = \{\alpha_1\}$, or $\{\alpha_2\}$,
     (B_l, Σ) $(l \geq 3)$ with $\Sigma = \{\alpha_1\}$,
  3. (C_l, Σ) $(l \geq 3)$ with $\Sigma = \{\alpha_1\}$,
  4. (D_4, Σ) with $\Sigma = \{\alpha_1\}$, $\{\alpha_3\}$, or $\{\alpha_4\}$,
     (D_l, Σ) $(l \geq 5)$ with $\Sigma = \{\alpha_1\}$.
- These cases correspond exactly to P^ℓ, Q^ℓ, $F_{1,\ell}(\mathbb{C}^{\ell+1})$.
- Note however, that even in these cases the corresponding rational homogeneous variety may or may not be rigid depending on the representation V. The complete classification of representations is unknown.

- Construction of moving frame works also in cases when G is an infinite-dimensional transitive Lie pseudo-group, and $\mathfrak{g}$ is a graded Lie algebra associated with an infinite-dimensional transitive Lie algebra of vector fields.
- Let $\mathfrak{g}$ be the Lie algebra of (polynomial) vector fields on $\mathbb{R}^2$ lifted to $J^2(\mathbb{R},\mathbb{R})$, and let N be a submanifold in J^2 transversal to the fibers of the projection $J^2 \to J^1$. In other words, N is a scalar second-order ODE viewed up to point transformations.
- In this case $\mathfrak{g}_-$ is a 4-dim nilpotent Lie algebra (= the symbol of the contact distribution on $J^2(\mathbb{R},\mathbb{R})$):

  $$[X, Y_1] = Y_2, [X, Y_2] = Y_3; \quad \deg X = -1, \deg Y_i = -i.$$

  One can show that transversality of N to the fibers of $J^2 \to J^1$ implies that N has a constant symbol $\mathfrak{g}_- = \langle X, Y_2, Y_3 \rangle$.
- The extrinsic prolongation $\mathrm{Prol}(\mathfrak{n})$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ (not surprisingly). Finally, one can prove that $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{sl}(3,\mathbb{R}))$ is isomorphic to $H^2(\mathfrak{n}, \mathfrak{sl}(3,\mathbb{R}))$. However, the gradings are different!